# Geometry of Root Lattices and Quantum Invariants

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## Motivation:

Relate q. invts of $K$ to topology of $S^3 \setminus K$.

## Specifically:

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{c}\text{q-holonom rels between} \\ \text{quant. invts of } K \\ \text{(recursive ideal)}\end{array} \arrow[dl, dash, "sl(2)"'] \arrow[dr, dash] & \\
\begin{array}{c}\text{skein module of } S^3 \setminus K \\ \text{(orthogonal ideal)}\end{array} \arrow[rr, dash, "sl(2)"'] & & \begin{array}{c}\text{Reps of } \pi_1(K) \\ \text{A-poly}\end{array}
\end{tikzcd}
$$

**Goal:** do it for other Lie algebras.

Goal: Geometric interpretation of q-holonomic relations between quantum invariants.

## Reshetikhin-Turaev invariants

$\mathfrak{g}$ = simple Lie alg. / $\mathbb{C}$

$V$ = f. dim rep of $\mathfrak{g}$

$K$ = knot in $S^3$

$$J_{\mathfrak{g},V}(K) \in \mathbb{Z}[q^{\pm\frac{1}{2D}}],$$

$D$ = det. of Cartan matrix of $\mathfrak{g}$,

e.g. $D(sl(n)) = n$.

# Representations of $\mathfrak{g}$

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra

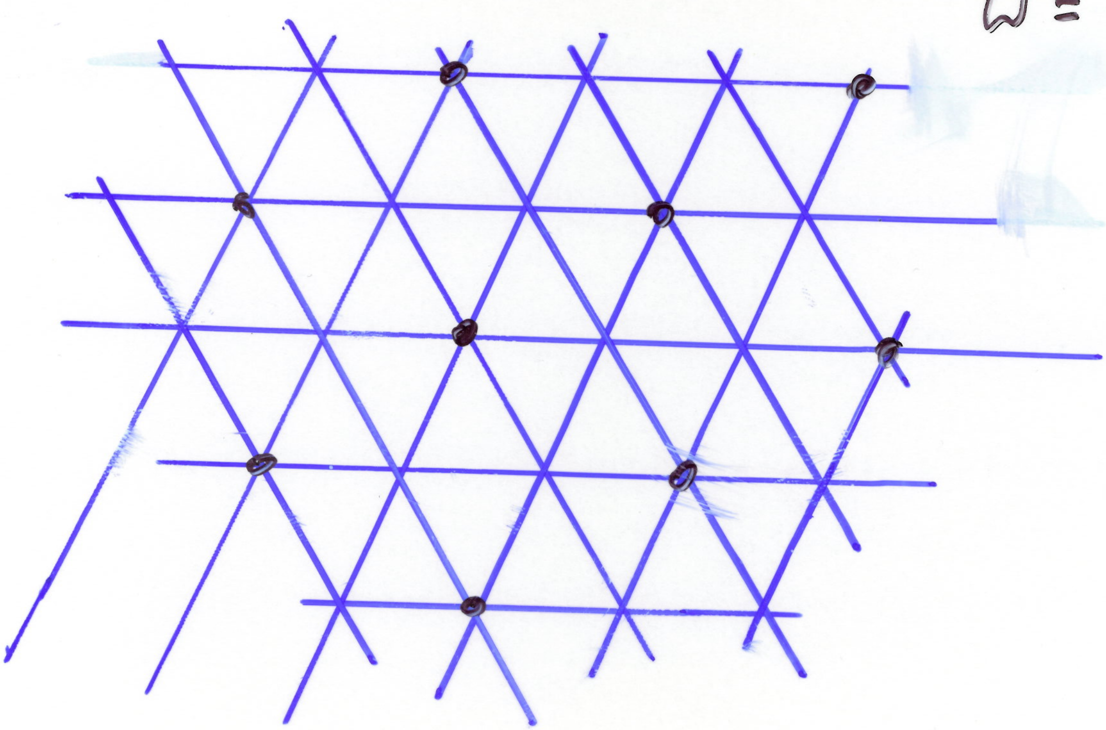
$$\underbrace{\Lambda'}_{\text{root lattice}} \subset \underbrace{\Lambda}_{\text{weight lattice}} \subset \mathfrak{h}^*$$

Weyl group acts on $\mathfrak{h}^*$, $\Lambda, \Lambda'$ are preserved.

$\mathfrak{g} = sl(2)$ $\qquad W = \mathbb{Z}/2$


$\mathfrak{g} = sl(3)$

$W = S_3$



$$\boxed{\text{Irred. reps of } \mathfrak{g} \xleftrightarrow{1-1} \Lambda/W}$$

For fixed $\mathfrak{g}$, $K$

$$J_{\mathfrak{g},K} : \Lambda/W \longrightarrow \mathbb{Z}[q^{\pm 1/2D}]$$

$W$ acts on the set of functions

$$F(\Lambda, \mathbb{Z}[q^{\pm 1/2D}]) \text{ and}$$

$$J_{\mathfrak{g},K} \in F(\Lambda, \mathbb{Z}[q^{\pm 1/2D}])^W.$$

Two families of operators on

$F(\Lambda, \mathbb{Z}[q^{\pm 1/2D}])$:

- $E_\alpha f(v) = f(v+\alpha)$ for any $\alpha, v \in \Lambda$
- $Q_\alpha f(v) = q^{\langle \alpha, v \rangle} f(v)$ for $v \in \Lambda$, $\alpha \in \Lambda^*$,

where $\langle \alpha, \beta \rangle = \dfrac{2B(\alpha,\beta)}{B(\alpha_0,\alpha_0)}$

$B(\cdot,\cdot)$ Killing form

$\alpha_0$ = any short root.

Def

$$\mathbb{A}_{\mathfrak{g}} = \mathbb{C}[q^{\pm 1/2D}]\langle E_\alpha, \alpha \in \Lambda, Q_\beta, \beta \in \Lambda' \rangle$$

<u>q-Weyl algebra of $\mathfrak{g}$</u>

$\mathbb{A}_{\mathfrak{g}} \subset \mathrm{End}(F(\Lambda, \mathbb{C}[q^{\pm 1/2D}]))$

$W$ acts on $\mathbb{A}_{\mathfrak{g}}$

$\mathbb{A}_{\mathfrak{g}}^{W}$ = <u>invariant q-Weyl alg. of $\mathfrak{g}$.</u>

## Examples:

$\underline{\mathfrak{g} = sl_2}$ $\quad A_{\mathfrak{g}} = \mathbb{C}[q^{\pm 1/4}]\langle E^{\pm 1}, Q^{\pm 1}\rangle \Big/ EQ = qQE$

$W = \mathbb{Z}/2 = \langle \tau \rangle \qquad \tau(E) = E^{-1}$
$\qquad\qquad\qquad\qquad \tau(Q) = Q^{-1}$

$\underline{\mathfrak{g} = sl_n}$

$$A_{\mathfrak{g}} = \mathbb{C}[q^{\pm 1/2n}]\langle E_1, \dots, E_n, Q_1, \dots Q_n\rangle \Big/ \begin{array}{l} E_i E_j = E_j E_i \\ Q_i Q_j = Q_j Q_i \\ \prod_{i=1}^n E_i = 1, \ \prod_{i=1}^n Q_i = 1 \\ Q_j E_k = q E_k Q_j \quad j \neq k \\ E_i Q_i = q^{n-1} Q_i E_i \end{array}$$

$W = S_n$ acts on $A_{\mathfrak{g}}$ $\qquad \sigma : E_i \to E_{\sigma(i)}$
$\qquad\qquad\qquad\qquad\qquad Q_i \to Q_{\sigma(i)}$

$\mathfrak{g}$-recursive ideal of $K$

$$I_{\mathfrak{g},K} = \{ P : P\, J_{\mathfrak{g},K} = 0 \} \lhd \mathbb{A}_{\mathfrak{g}}$$

Each $P \in I_{\mathfrak{g},K}$ is a recursive relation on $J_{\mathfrak{g},K}$

Thm (Garoufalidis-Le)

For $\mathfrak{g} \neq G_2$, $J_{\mathfrak{g},K} \in F(\Lambda, \mathbb{C}[q^{\pm 1/2D}])$ is uniquely determined by recursive rels in $I_{\mathfrak{g},K}$ and a finite number of initial values of $J_{\mathfrak{g},K}$

Thm (S.)

For $\mathfrak{g} \neq G_2$, $J_{\mathfrak{g},K}$ is uniquely determined by recursive rels in $I^W_{\mathfrak{g},K}$ and a fin. number of values of $J_{\mathfrak{g},K}$.

$$I^W_{\mathfrak{g},K} = I_{\mathfrak{g},K} \cap \mathbb{A}^W_{\mathfrak{g}} \lhd \mathbb{A}^W_{\mathfrak{g}}$$

invariant $\mathfrak{g}$-recursive ideal of $K$

**Claim:** $I^{w}_{g,K} \triangleleft A^{w}_{g}$ is more closely related to topology of $S^3 \setminus K$ then $I_{g,K} \triangleleft A_g$.

# Character Varieties

$\Gamma$ = discrete group, $G$ = alg. group / $\mathbb{C}$

Then $G$-character variety of $\Gamma$ is

$$X_G(\Gamma) \overset{\text{def}}{=} \mathrm{Hom}(\Gamma, G)/\!\!/G$$

$$\mathrm{Hom}(\Gamma, G)/\!\!/G = \mathrm{Hom}(\Gamma, G)/G \Big/ \begin{matrix} \rho_1 = \rho_2 \text{ if} \\ \rho_1 \in \overline{\rho_2} \end{matrix}$$

It is an algebraic set.

Thm (S.) If $G = SL(n), SO(n), Sp(n)$, $\mathfrak{g}$ = Lie alg. of $G$, then $A_{\mathfrak{g}}^{\omega}$ is a deformquant. of $\mathbb{C}[X_G(\mathbb{Z}^2)]$.

$$A_{\mathfrak{g}}^{\omega} \otimes_{q \to 1} \mathbb{C} = \mathbb{C}[X_G(\mathbb{Z}^2)].$$

## What about $I_{\mathfrak{g},k}^{\omega} \triangleleft A_{\mathfrak{g}}^{\omega}$ ?

We have $\varepsilon : A_{\mathfrak{g}}^{\omega} \xrightarrow[q \to 1]{} \mathbb{C}[X_G(\mathbb{Z}^2)]$

$\varepsilon(I_{\mathfrak{g},k}^{\omega})$ is an ideal in $\mathbb{C}[X_G(\mathbb{Z}^2)]$.

Let $M_K = \overline{S^3 \setminus K}$, $\partial M_K = T$.

$\partial M_K \hookrightarrow M_K$ induces $X_G(\pi_1(M_K)) \to X_G(\underset{\pi_1(T)}{\underset{\parallel}{\mathbb{Z}^2}})$

and $p_K : \mathbb{C}[X_G(\mathbb{Z}^2)] \to \mathbb{C}[X_G(\pi_1(M_K))]$.

$\operatorname{Ker} p_K =$ "<u>A-ideal of $K$</u>" $\triangleleft\ \mathbb{C}[X_G(\mathbb{Z}^2)]$.

$$\boxed{\text{Conj. } \varepsilon(I^W_{\mathfrak{g},K}) = \operatorname{Ker} p_K \text{ for every } \mathfrak{g}, K}$$

By T. Le, it holds for $\mathfrak{g} = sl_2$, $K =$ 2-bridge knots

# Skein Modules

$M$ = oriented 3-mfld

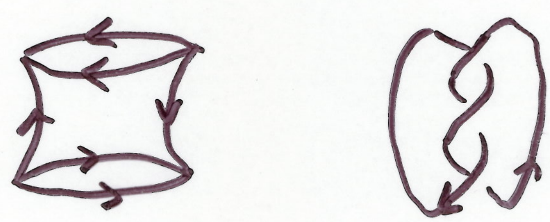
$sl(2)$-skein module of $M$

$$S_{sl(2)}(M) = \frac{\mathbb{C}[q^{\pm 1/2}]\{\text{fr. links in } M\}}{\quad = q^{1/2}\,)(\, + q^{-1/2}\,\asymp \qquad L \cup \bigcirc = (-q-q^{-1})L}$$

Let $\mathfrak{g} = sl(n)$

$n$-webs in $M$ = $n$-valent oriented graphs in $M$ whose vertices are sources or sinks

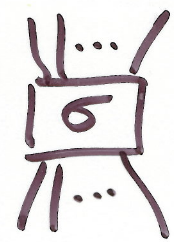
Eg. 3-web



$$S_{SL(n)}(M) = \frac{\mathbb{C}[q^{\pm 1/2n}]\{n\text{-webs in } M\}}{\phantom{x}}$$

$q^n$  $- q^{-n}$  $= (q-q^{-1})$ 

 $= \dfrac{q^n - q^{-n}}{q - q^{-1}}$

 $= \sum_{\sigma \in S_n} (-q^{n-1})^{\ell(\sigma)}$ 


unique "simplest"
positive braid representing
$\sigma \in S_n$.

Claim: $S_{sl(n)}(M)$ has all desired properties of $sl(n)$-skein module. For example,

Thm (S.)

$$S_{sl(n)}(M) \otimes_{q \to 1} \mathbb{C} = \mathbb{C}[X_{SL(n)}(\pi_1(M))].$$

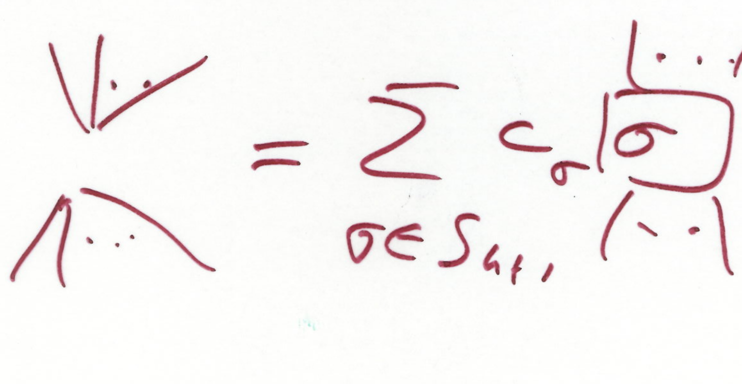
For rank 2 Lie algebras, $S_{\mathfrak{g}}(M)$ can be defined by Kuperberg's spiders.

Open: Define $S_{\mathfrak{g}}(M)$ for every $\mathfrak{g}$.

From now on, let $\mathfrak{g} = sl(n)$ or rank 2

$(o(5) = sp(4),\ \mathfrak{g}_2)$.

$$S_{so(n)}(M) = \mathbb{C}[(n+1)\text{-valent graphs}]$$



If $M = F \times I$, $F$ = surface, then

$S_{\mathfrak{g}}(M)$ has multiplicative structure.

$S_{\mathfrak{g}}(F)$ = "the $\mathfrak{g}$-skein algebra of $F$".

Thm (S.)

For $\mathfrak{g} = sl(n)$, $S_{\mathfrak{g}}(T \times I)$ is a deformation-quantization of $\mathbb{C}[X_{SL(n)}(\mathbb{Z}^2)]$.

Recall that $\mathbb{A}_{\mathfrak{g}}^{W}$ is a deform-quant. of $\mathbb{C}[X_G(\mathbb{Z}^2)]$ as well.

Conj

$$\mathbb{A}_{\mathfrak{g}}^{W} \simeq S_{\mathfrak{g}}(T \times I) \text{ for every } \mathfrak{g}.$$

More specifically, if $\gamma = (l,m)$-loop in $T$ then

$$\phi : S_{SL(n)}(T \times I) \longrightarrow \mathbb{A}_{sl(n)}^{W}$$

$$\phi(\gamma) = \sum_i q^{-\frac{lm}{n}} E_i^{l} Q_i^{m}$$

is an isom of $\mathbb{C}[q^{\pm 1/2n}]$-algebras.

By Frohman-Gelca, Conj. holds for $sl(2)$.

Reps of $\mathfrak{g}$, together with $\oplus$, $\otimes$

Grothendieck completion $\Rightarrow$ Rep $\mathfrak{g}$

representation ring of $\mathfrak{g}$

$$\operatorname{Rep}\mathfrak{g} = (\mathbb{C}\Lambda)^W$$

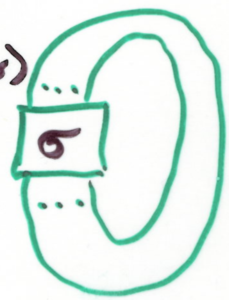
**Thm** (S.) If $A$ = annulus, $\mathfrak{g} = sl(n)$

$$S_{\mathfrak{g}}(A \times I) \simeq \operatorname{Rep}(\mathfrak{g}) \otimes \mathbb{C}[q^{\pm 1/2n}] = (\mathbb{C}[q^{\pm 1/2n}]\Lambda)^W$$

Furthermore, we have the following iso

$$\psi : S_{\mathfrak{g}}(A \times I) \to (\mathbb{C}[q^{\pm 1/2n}]\Lambda)^W$$

$$\sum_{\sigma \in S_k} \left(-q^{\frac{1-n}{n}}\right)^{\ell(\sigma)}$$




$\longrightarrow \sum$ weights of $\Lambda^k \mathbb{C}^n$ for $\mathfrak{g} = sl(n)$.

$$\underset{\overset{?}{\cong}}{S_g(T\times I)} \text{ acts on } S_g(\overset{\overset{D^2\times S^1}{\cong}}{A\times I})$$

$$\overset{\phantom{x}}{\cong}\quad\overset{?}{\phantom{x}}\qquad\qquad\qquad\qquad \cong$$

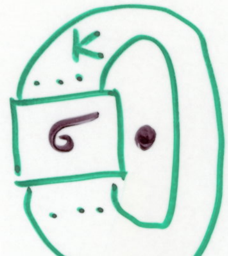
$$A_{\mathfrak{g}}^{W} \text{ acts on } (\mathbb{C}[q^{\pm 1/2n}]\Lambda)^W$$

<u>Conj</u> "$S = q\mathcal{H}$"

$$\begin{tikzcd}
S_g(T\times I) \arrow[d, "\phi"] & \times & S_g(\overset{\overset{D^2\times S^1}{\shortparallel}}{A\times I}) \arrow[d, "\psi"] \arrow[r] & S_g(A\times I) \arrow[d, "\psi'"] \\
A_{\mathfrak{g}}^{W} & \times & (\mathbb{C}[q^{\pm 1/2n}]\Lambda)^W \arrow[r] & (\mathbb{C}[q^{\pm 1/2n}]\Lambda)^W
\end{tikzcd}$$

where $\psi: S_g(A\times I) \to (\mathbb{C}[q^{\pm 1/2n}]\Lambda)^W$

for $\mathfrak{g} = sl(n)$

$$\sum_{\sigma\in S_k} (-q^{n-1})^{\ell(\sigma)}$$  $\to \sum$ weights of $\Lambda^k\mathbb{C}^n$

for even $k = 0, \dots, n-1$

(We can prove that $\psi$ is an isom.)

Conj. holds for $SL(2)$.

Let $K \subset S^3$, $T = \partial N(K)$,

$$\begin{array}{ccc} S_{\mathfrak{g}}(T\times I) \times S_{\mathfrak{g}}(D^2\times S^1) & \to & S_{\mathfrak{g}}(D^2\times S^1) \\ \quad\quad\quad\quad \shortparallel & & \downarrow \\ \quad\quad\quad\quad N(K) & & S_{\mathfrak{g}}(S^3) = \mathbb{C}[q^{\pm}] \end{array}$$

Denote this pairing by $\langle \cdot, \cdot \rangle$.

$$O_{\mathfrak{g},K} = \{ x \in S_{\mathfrak{g}}(T\times I) : \forall_{y \in S_{\mathfrak{g}}(D^2\times S^1)} \ \langle x, y\rangle = 0 \}$$

$O_{\mathfrak{g},K} \triangleleft S_{\mathfrak{g}}(T\times I)$ is

the $\mathfrak{g}$-orthogonal ideal of $K$

Thm (S.)

If $S = qtt$ conjecture holds then

$\varphi : S_{\mathfrak{g}}(T\times I) \to A^W_{\mathfrak{g}}$ identifies

$O_{\mathfrak{g},K}$ with $I^W_{\mathfrak{g},K}$.

Additionally, $\underbrace{\varepsilon}_{\text{evaluation } q=1}(I^W_{\mathfrak{g},K}) \subset \operatorname{Ker} f_K$. for $sl(n)$ "A-ideal"

$(f_K : \mathbb{C}[X_G(\mathbb{Z}^2)] \to \mathbb{C}[X_G(\pi_1(S^3 \setminus K))])$

Comments about

Conj $\quad S_{\mathfrak{g}}(T\times I) \simeq A_{\mathfrak{g}}^{W}$

Holds for $\mathfrak{g} = sl(2)$

Basis of $S_{sl_2}(T\times I) = \left\{ \begin{array}{l} \text{systems of non-inters} \\ \text{curves in } T \end{array} \right\}$

$$\| $$

$$\{ (p,q),\ q\geq 0,\ p\in\mathbb{Z} \}$$

$$\& \ (p,q)\neq(0,0).$$

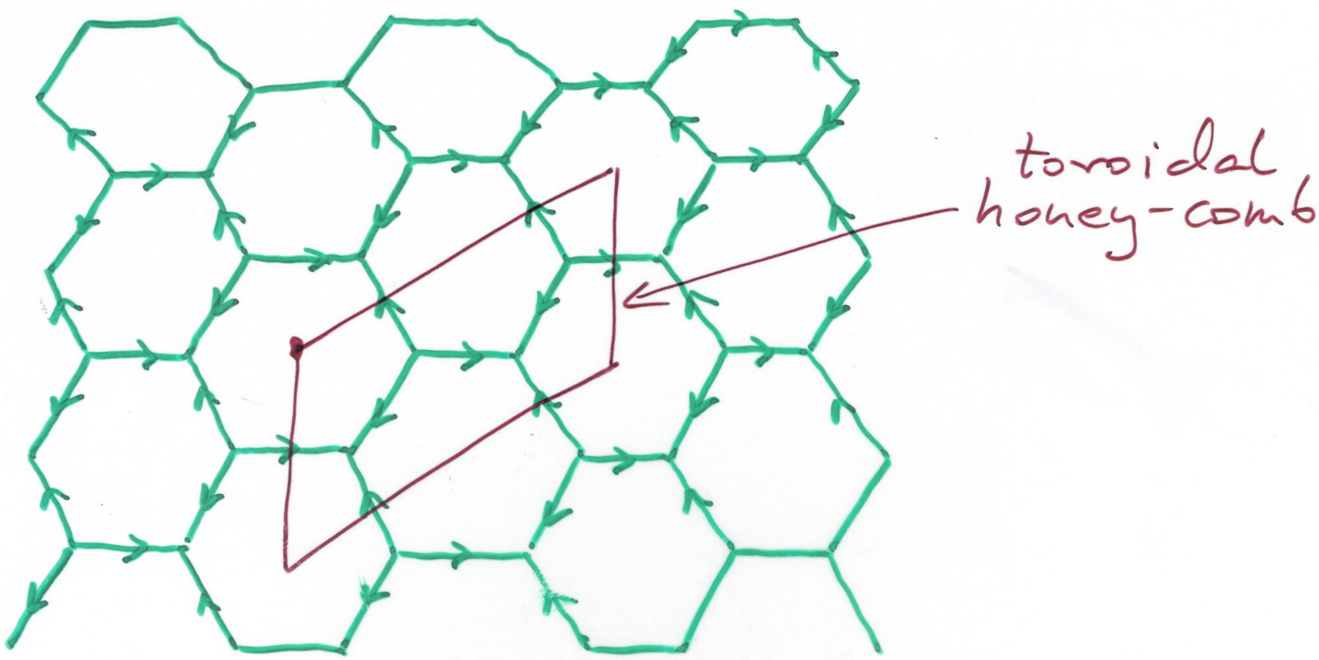
$$S_{sl(2)}(T\times I) = \left( \mathbb{C}[q^{\pm 1/4}]\langle E^{\pm 1}, Q^{\pm 1}\rangle \big/ EQ = qQE \right)^{\tau}$$

$$\tau : E \to E^{-1}$$
$$Q \to Q^{-1}$$

$\mathfrak{g} = sl(3)$

$\mathfrak{g} = sl(3)$

$$S_{\mathfrak{g}}(T\times I) = \mathbb{C}[q^{\pm 1/6}]\left\{ \begin{array}{c}\text{3-valent graphs} \\ \text{in } T\end{array} \right\} \Big/ \begin{array}{c}\text{skein} \\ \text{rels}\end{array}$$




**Thm** (S.)

$S_{sl(3)}(T\times I)$ has a basis composed of

- systems of non-intersecting curves in $T$
- toroidal honeycombs

**Conj.** $S_{SL_3}(T \times I) \simeq \mathbb{A}^{W}_{SL(3)}$

$$\mathbb{A}^{W}_{SL(3)} = \left( \mathbb{C}[q^{\pm 1/6}] \langle E_i, Q_i, \; i=1,2,3 \rangle \big/ \text{rels} \right)^{S_3}$$

**Prop**

$\mathbb{A}^{W}_{SL(3)}$ has a basis composed of

- $\displaystyle\sum_{i=1}^{3} E_i^{l} Q_i^{m}$
- $\displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} E_i^{a} E_j^{b} Q_i^{c} Q_j^{d}$, where $(a,c), (b,d) \in \mathbb{Z}^2$ are lin. indep and defined up to Weyl group action