A functorial extension of the Le-Murakami-Ohtsuki invariant and some applications

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1. Kontsevich integral and LMO invariant
2. Kontsevich-LMO functor
   - domain
   - codomain
   - construction
3. Properties and applications

1.1. Finite-type invariants of knots / links

$\mathcal{K}$ = the set of oriented polygonal knots up to isotopy

Vassiliev filtration

$$V_0 \supset V_1 \supset \ldots \supset V_n \supset \ldots$$
$$V_0 = \mathbb{Z}\{\mathcal{K}\}$$
$$V_n = \mathbb{Z}\{[K^m] \mid m \geq n\}$$

where $[K^m]$ = desingularization of K^m, a singular knot with m transverse double points, obtained by recursively applying

Definition $v : \mathbb{Z}\{\mathcal{K}\} \to \mathbb{C}$ is a finite-type invariant of degree $\leq n$ if $v([K^m]) = 0$, $\forall K^m$, $\forall m > n$

$\mathcal{A}(S^1)$ = algebra of chord diagrams on  $_{S^1}$ / (4T) relation

$\Downarrow$

$$\varphi_n : \mathcal{A}_n(S^1) \longrightarrow V_n / V_{n+1}, \quad \forall n$$

weld the points of S^1 in the indicated order, and desingularize.

Theorem (Kontsevich, 1991, $\mathbb{C}$; ext./$\mathbb{Q}$ by Le-Murakami 1995)
There exists an invariant $Z : \mathcal{K} \to \mathcal{A}(S^1) \otimes k$ which preserves the filtrations $Z(V_n \otimes k) \subseteq \mathcal{A}_{\geq n} \otimes k, \forall n$ and induces isomorphisms $Z_n : V_n / V_{n+1} \otimes k \xrightarrow{\simeq} \mathcal{A}_n \otimes k$. Moreover, $Z_n = (\varphi_n \otimes k)^{-1}$!

$$\nu = Z(\text{unknot}) = \chi\left(\exp\left(\sum_{n=1}^{\infty} b_{2n} w_{2n}\right)\right)$$

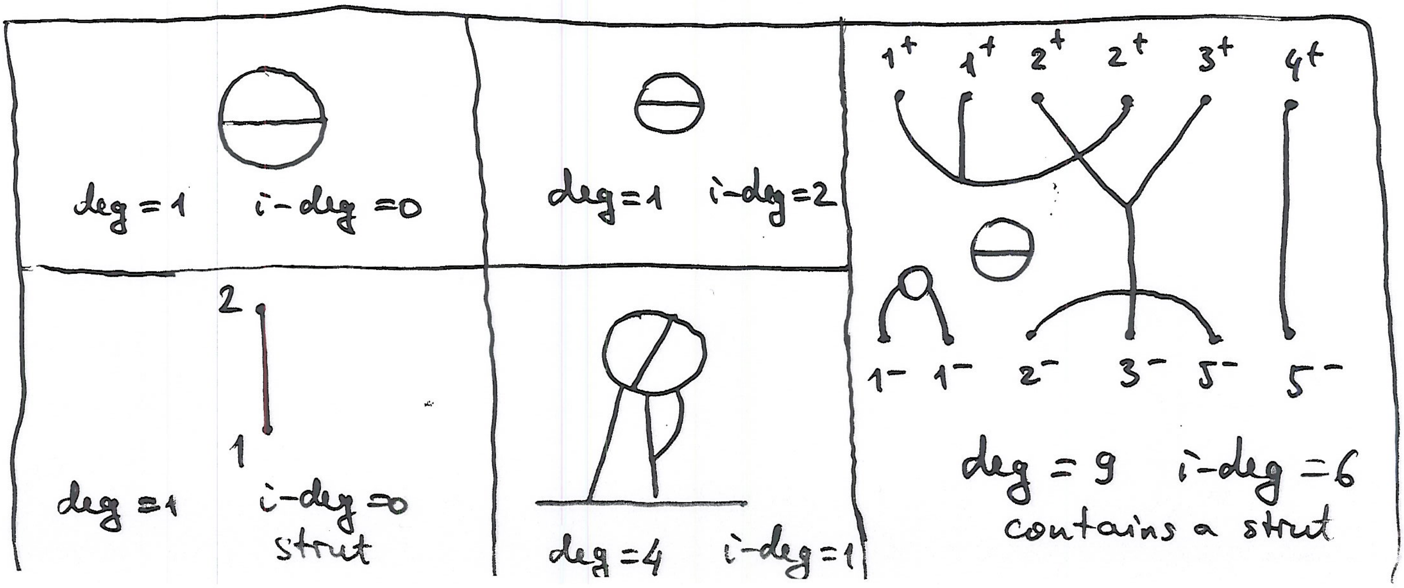
where $\displaystyle\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \log \frac{\sinh \frac{x}{2}}{\frac{x}{2}}$

$$b_2 = \frac{1}{48}, \quad b_4 = -\frac{1}{5760}, \quad b_6 = \frac{1}{362880}$$

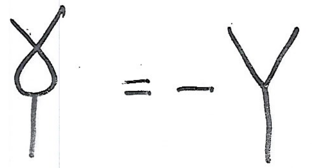
$w_2 =$ , $w_4 =$ , $w_6 =$ 

χ = average of all ways of gluing a Jacobi diagram on $2n$ points to S^1

1.2. Jacobi diagrams

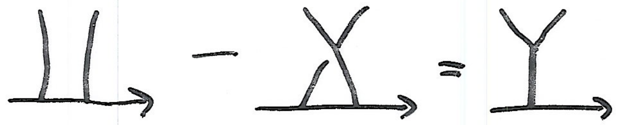


X = 1-manifold
C = a set
D = a Jacobi diagram based on (X, C) is a uni-trivalent vertex-oriented graph whose external vertices are either embedded in X, or colored by elements of C, up to data-preserving homeo



AS

IHX



STU

$$\mathcal{A}(X, C) := \mathbb{Q}\left\{\frac{\text{Jacobi diagrams based on } (X, C)}{\text{AS, IHX, STU}}\right\}^{\wedge}$$

$\wedge$ = degree completion

$$\mathcal{A}(C) \nearrow \mathcal{A}^{s}(C) \quad \text{strut-part}$$
$$\mathcal{A}(C) \searrow \mathcal{A}^{Y}(C) \quad \text{Y-part}$$

1.3. Le – J. Murakami – Ohtsuki invariant

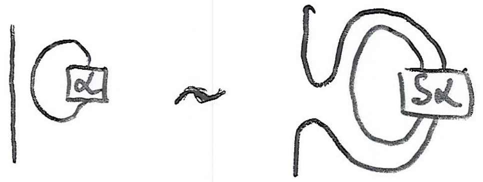
Z^{LMO} : M (closed 3-manifold) $\longmapsto$ group like element of $\mathcal{A}(\emptyset)$, a Hopf algebra

- multiplication = disjoint union
- $\Delta(D) = \sum_{D = D' \sqcup D''} D' \otimes D''$
- primitives = degree completion of the subspace $\mathcal{A}^c(\emptyset)$ generated by connected non-empty Jacobi diagrams

$\check{Z}(L) := Z(L) \otimes \nu^{|L|}$

$\iota_n : \mathcal{A}(\bigsqcup_{|L|} S^1) \to \mathcal{A}(\emptyset)$

s.t. $\deg_{\leq n}(\iota_n \check{Z}(L))$ invariant under Kirby-2 move



$M \cong S^3_L$

$$\Omega_n(M) = \deg_n \left(\frac{\iota_n \check{Z}(L)}{c_+^{\sigma_+} c_-^{\sigma_-}} \right)$$

where $(\sigma_+, \sigma_-) = \operatorname{sign} lk(L)$

$$c_+ = \lim_n (-1)^n \deg_{\leq n}(\iota_n \check{Z}(O^{+1}))$$

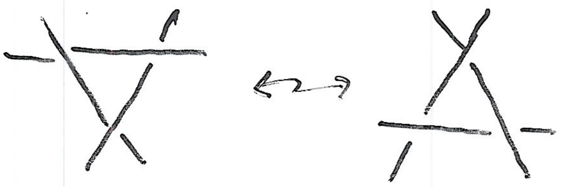
$$c_- = \lim_n \deg_{\leq n}(\iota_n \check{Z}(O^{-1}))$$

QHS $\quad Z^{LMO}(M) = \sum_{n \geq 0} \frac{1}{|H_1(M,\mathbb{Z})|^n} \cdot \Omega_n(M)$

1.4. Finite-type invariants of 3-dimensional manifolds

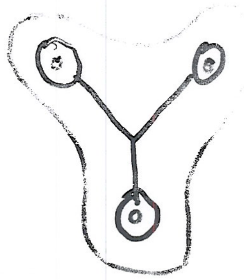
$\mathcal{F}_0 \supset \mathcal{F}_1 = \mathcal{F}_2 \supset \mathcal{F}_3 = \mathcal{F}_4 \supset \dots$

$\mathcal{F}_1$ is induced by formal differences $M_L - M_{L'}$ s.t. L and L' are related by a Δ-move $\Leftrightarrow$ $\Leftrightarrow$ a Borromean linking $\Leftrightarrow$ a surgery on a Y_1-graph

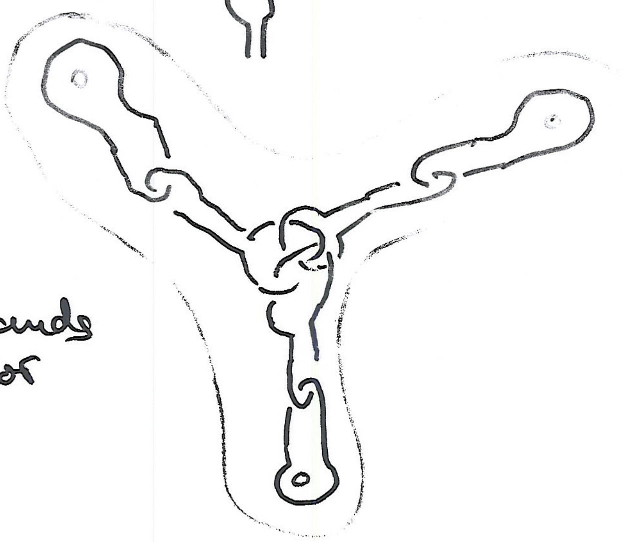


Δ - move

Borromean linking

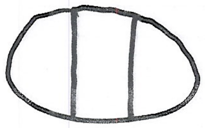


stands for



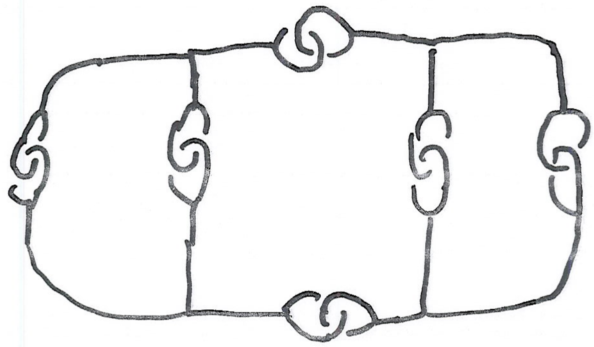
surgery on a Y_1 graph

$$\mathcal{F}_n = \Big\langle \sum_{G' \subset G} (-1)^{|G'|} M_{G'} \;\Big|\; \begin{array}{l} M \in \text{a fixed } Y_1\text{-equivalence class} \\ G \text{ a graph clasper of i-deg } n \end{array} \Big\rangle$$



i-deg = 4

$\xmapsto{W}$



Z^{LMO} is a universal finite type invariant for homology 3-spheres (a Y_1-equivalence class):

$$\mathcal{A}(\emptyset)_n \xrightarrow[\cong]{W} \mathcal{F}_{2n}/\mathcal{F}_{2n+2} \xrightarrow[\cong]{Z^{LMO}} \mathcal{A}(\emptyset)_n.$$

- $Z^{LMO}(M_1 \# M_2) = Z^{LMO}(M_1) \cdot Z^{LMO}(M_2)$
- $Z^{lmo}(M) = 1 + \dfrac{(-1)^{\operatorname{rank} H_1(M,\mathbb{Z})} \cdot \lambda(M)}{2} \ominus + h.o.t$

where $\lambda(M)$ = Casson-Walker-Lescop invariant

- $Z^{LMO}(M)$ classifies Seifert-fibered $\mathbb{Z}HS$
- $Z^{lmo} = 0$ if $\operatorname{rank} H_1(M,\mathbb{Z}) \geq 4$

2. Kontsevich-LMO functor

Theorem. [CHM] The Kontsevich-LMO invariant admits an extension to a tensor-preserving functor

$$\tilde{Z} : \mathcal{L}Cob_q \to {}^{ts}\mathcal{A}$$

from the category of Lagrangian q-cobordisms between 1-punctured surfaces to the category of top-substantial Jacobi diagrams.

Properties (simple observations and difficult facts):

1. $\tilde{Z}$ descends to a functor $\tilde{Z} : \widehat{\mathcal{L}Cob_q} \to {}^{ts}\hat{\mathcal{A}}$ from the category of Lagrangian q-cobordisms between closed surfaces [J. Murakami, T. Ohtsuki, 1996; D.C., T. Le, 2005(2008)]. Moreover, the latter does not depend on the choice of Drinfeld associator.

2. $\tilde{Z} = \underset{\text{strut part}}{\tilde{Z}^s} \sqcup \underset{\text{internal part}}{\tilde{Z}^Y}$

3. $\tilde{Z}^Y$ is a universal FTI for Lagrangian cobordisms.

4. Induces a monoid homomorphism

$$\tilde{Z}^Y : (\mathcal{C}yl(F_g), \circ) \to (\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-), \star)$$

5. The combinatorial multiplication $\star$ admits a simple description.

6. Truncations of $\tilde{Z}^Y$ induce finite-dimensional representations of the Torelli group

7. [G. Massuyeau; G. Massuyeau - K. Habiro] The tree-part $\tilde{Z}^{Y,t}$ is a (very) powerful invariant of the Torelli group / homology (Torelli) cylinders.

Major problems:

1. Topological interpretation of the induced representation of the Torelli group

2. Impossible (yet) to relate directly to the Reshetikhin - Turaev TQFT

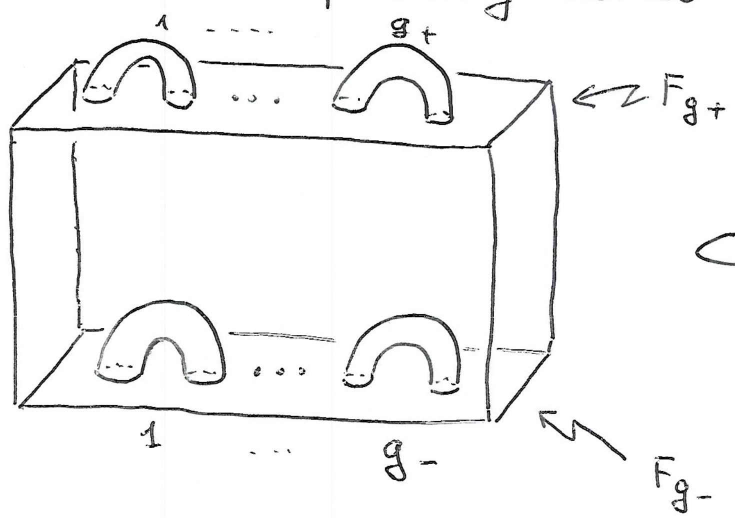
3. Applications of the non-tree part of $\tilde{Z}^Y$.

2.1. The domain of $\tilde{Z}$: Lagrangian cobordisms

$F_g := \Sigma_{g,1}$

<u>Def</u> (M, m) is a <u>cobordism</u> from F_{g+} to F_{g-}
if M is a compact connected oriented 3-manifold,
$m : \partial C^{g+}_{g-} \to \partial M$ orientation-preserving homeo onto,
where

$C^{g+}_{g-} =$



$\subset S^3$

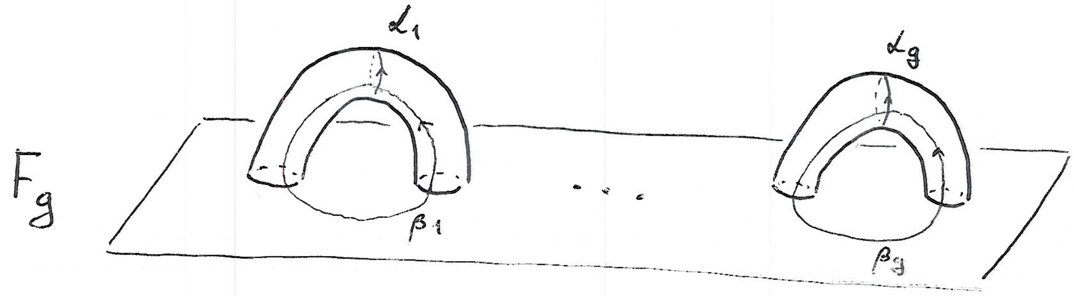
is a cube in S^3 with g_+ handles and g_- tunnels
fixed once and for all $(\forall g_+, g_- \geq 0)$.

<u>Obs</u> m induces parametrizations $m_\pm : F_{g\pm} \to M$ of
<u>top</u> and <u>bottom</u> boundary components of M.

<u>Obs</u> Cobordisms are considered up to equivalence
$(M, m) \sim (M', m') \overset{def}{\iff} \exists f : M \to M'$ orientation-preserving homeo s.t. $f \circ m_\pm = m'_\pm$

Cob = structure of a monoidal category
objects : $g \geq 0$
morphisms : equivalence classes of cobordisms

$(M, m) \circ (N, n) = \boxed{\begin{array}{c}(N, n) \\ \hline (M, m)\end{array}}$ $\qquad (M, m) \otimes (N, n) = \boxed{(M, m) \mid (N, n)}$



Denote $A_g := \ker\left(\text{incl}_* : H_1(F_g, \mathbb{Z}) \to H_1(C_0^g, \mathbb{Z})\right) = \langle \alpha_1, \dots, \alpha_g \rangle$ meridians

It is a Lagrangian subgroup of $H_1(F_g, \mathbb{Z})$

Obs Analogously, $B_g = \langle \beta_1, \dots, \beta_g \rangle$ longitudes

Def (M, m) is a <u>Lagrangian cobordism</u> if it satisfies conditions (1) and (2), or equivalently (1′) and (2

(1) $H_1(M) = m_{-*}(A_{g_-}) + m_{+*} H_1(F_{g_+})$

(2) $m_{+*}(A_{g_+}) \subset m_{-*}(A_{g_-})$ as subgroups of $H_1(M)$

(1′) $m_{-*} + m_{+*} : A_{g_-} \oplus B_{g_+} \to H_1(M)$ iso

Obs $M \cup_m (S^3 \setminus C_{g_-}^{g_+})$ is a homology 3-sphere

Obs coefficients in $\mathbb{Z}$ or $\mathbb{Q}$ (fix)

Def (M, m) is a <u>doubly Lagrangian cobordism</u> if it is A_g-Lagrangian from F_{g_+} to F_{g_-} and B_g-Lagrangian from F_{g_-} to F_{g_+}, or equivalently (1′), (2) and (2′)

(2′) $m_{-*}(B_{g_-}) \subset m_{+*}(B_{g_+})$ as subgroups of $H_1(M)$

Def (M, m) is a <u>special Lagrangian cobordism</u> if it is Lagrangian and $C_0^{g_-} \circ M = C_0^{g_+}$

Mapping Cylinder construction:

$$MCG(g,1) \rightarrowtail Cob(g,g)$$

$$h \longmapsto \left(F_g \times [-1,1],\ (Id \times \{-1\}) \cup (h \times 1) \right)$$

monoid homomorphism

Def (M,m) cobordism from F_g to F_g is a <u>homology cobordism</u> if $m_{\pm *}: H_1(F_\pm) \to H_1(M)$ are isomorphisms

Def (M,m) cobordism from F_g to F_g is a <u>homology cylinder</u> if $m_{\pm *}: H_1(F_\pm) \to H_1(M)$ are isomorphisms and $(m_+)_* = (m_-)_*$.

Obs Torelli group $\mathcal{I}(g,1) \hookrightarrow Cyl(F_g)$

homology cylinders (also called Torelli cylinders)

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doubly Lagrangian ${}^{d}\mathcal{L}Cob$

special Lagrangian ${}^{s}\mathcal{L}Cob$

$${}^{d}\mathcal{L}Cob,\ {}^{s}\mathcal{L}Cob \subset \mathcal{L}Cob \subset Cob$$

$\mathcal{L}Cob$: Lagrangian cobordisms

$h_*(A_g) = A_g$, $h_*(B_g) = B_g$: doubly-Lagrangian homeos

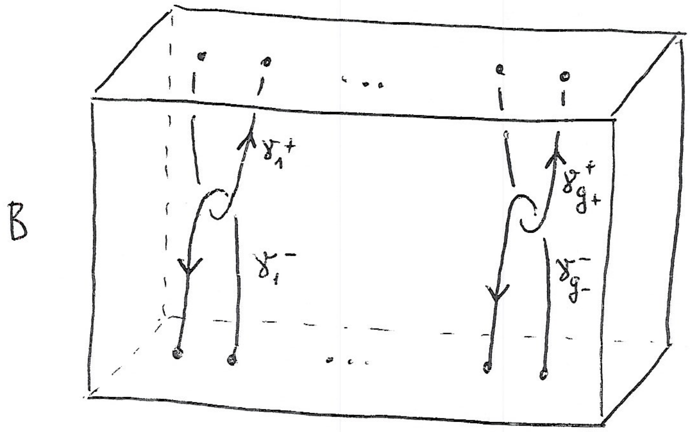
Lagrangian homeos extending to the full handlebody

doubly-Lagrangian homeos, Lagrangian homeos extending to the full handlebody $\subset$ Lagrangian homeos $\subset MCG(g,1)$

Lagrangian homeos: h s.t. $h_*: H_1(F_g,\mathbb{Z}) \to H_1(F_g,\mathbb{Z})$, $h_*(A_g) = A_g$

Def (B, γ) is a <u>bottom – top tangle</u> if
B is a cobordism from F_0 to F_0 and
γ is a framed oriented tangle with
components $\gamma_1^-, \dots, \gamma_{g-}^-, \gamma_1^+, \dots, \gamma_{g+}^+$
which begin and end in the bottom, respectively top surfaces F_0.

Example



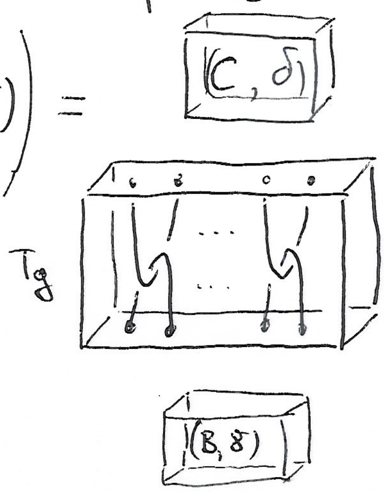
this particular example will be Id_g in category ${}_b^t\mathcal{T}$

Obs $(B, \gamma) \sim (B', \gamma')$ iff $\exists$ equiv. of cobordisms $B \to B'$ sending γ_i^- to $\gamma_i'^-$ and γ_j^+ to $\gamma_j'^+$

${}_b^t\mathcal{T}$ = category of bottom – top tangles : objects : $g \geq 0$
morphisms : bottom – top tangles

Composition rule : insert the particular tangle T_g and perform surgery on $2g$ link components

$$(B, \gamma) \circ (C, \delta) = \left((B \circ C)_{\text{surgery on } \gamma^+ \cup T_{g+} \cup \delta^-} , (\delta^+, \gamma^-) \right) =$$



Theorem There is a natural isomorphism of monoidal categories ${}_b^t\mathcal{T} \to \mathcal{C}ob$

$$\boxed{{}^t_b\mathcal{T} \xrightarrow{\ \cong\ } \mathcal{C}ob}$$

Alternative names:

(B,γ) s.t.
B = homology cube,
$lk(\gamma^+) = 0$, i.e.
$lk(\gamma) = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$
$\longleftrightarrow$ $(M,m) \in \mathcal{L}Cob$ — bottom tangles in homology handlebodies

(B,γ) s.t.
B = homolog cube,
$lk(\gamma) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$
$\longleftrightarrow$ $(M,m) \in {}^d\mathcal{L}Cob$ — bottom tangles with zero linking matrix in homology handlebodies

(B,γ) s.t.
$B = [-1,1]^3 = C_0^0$,
$\gamma^+ = \cup \ldots \cup$ = trivial tangle
$\longleftrightarrow$ $(M,m) \in {}^s\mathcal{L}Cob$ — bottom tangles in handlebodies

<u>Obs</u> $\left\{\begin{matrix}\text{Homology}\\ \text{cylinders}\end{matrix}\right\} \subset \left\{\begin{matrix}{}^d\text{Lagr.}\\ \text{cob.}\end{matrix}\right\} = \left\{\begin{matrix}\text{Lagr.}\\ \text{cob}\end{matrix}\right\} \cap \left\{\begin{matrix}\text{Homology}\\ \text{cobordisms}\end{matrix}\right\} \longleftrightarrow$ (B,γ) s.t. B = homology cube, $lk(\gamma) = \begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}$

<u>Thm</u> [Matveev + Murakami, Nakanishi] Two bottom-top tangles in homology cubes (B,γ) and (B',γ') are Y_1-equivalent $\Leftrightarrow$ they have the same linking matrix

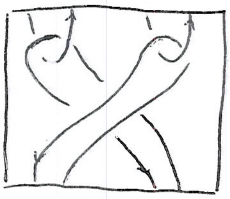
<u>Corollary</u> (M,m) and (M',m') Lagrangian cobordisms between F_{g+} and F_{g-} are Y_1-equivalent $\Leftrightarrow \exists$ iso $\psi : H_1(M) \to H_1(M')$ such that

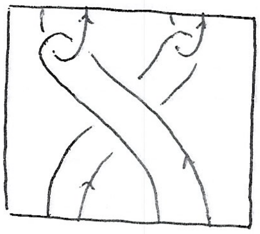
$$\begin{array}{ccccc} & & H_1(M) & & \\ & \nearrow^{m_{-*}} & & \nwarrow^{m_{+*}} & \\ H_1(F_{g-}) & & \psi\downarrow\cong & & H_1(F_{g+}) \\ & \searrow_{m'_{-*}} & & \swarrow_{m'_{+*}} & \\ & & H_1(M') & & \end{array}$$

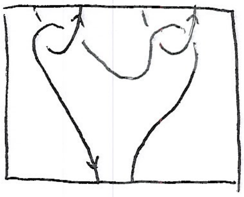
commutes

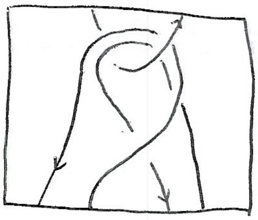
<u>Corollary</u> Any Lagrangian cobordism is Y_1-equivalent to a special Lagrangian cobordism.

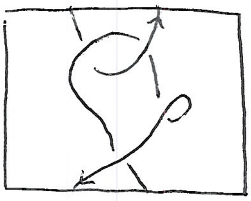
Generators

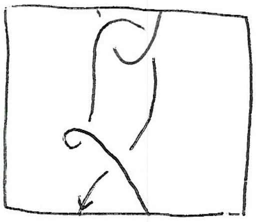
$\psi_{1,1} =$ 

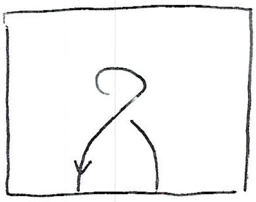
$\psi_{1,1}^{-1} =$ 

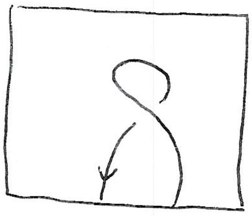
$\mu =$ 

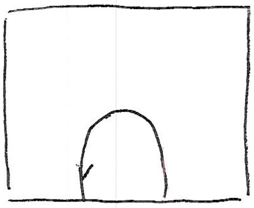
$\Delta =$ 

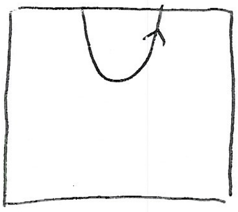
$S =$ 

$S^{-1} =$ 

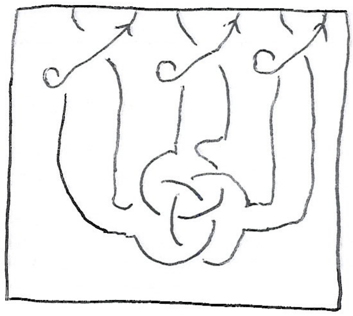
$v_+ =$ 

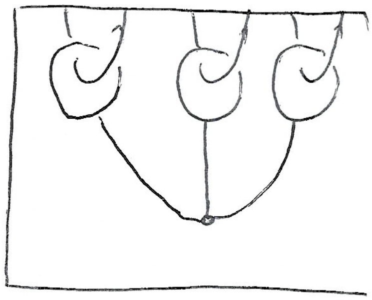
$v_- =$ 

$\eta =$ 

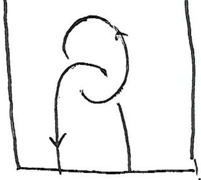
$\varepsilon =$ 

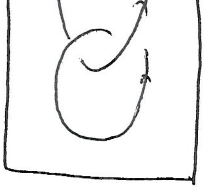
$^{s}\mathcal{L}Cob$

$Y =$ 

$=$ 

$\mathcal{L}Cob = \langle {}^{s}\mathcal{L}Cob, Y \rangle$

$=$ 

$m =$ 

$Cob = \langle {}^{s}\mathcal{L}Cob, e, m \rangle$

[Crane, Yetter]

[Kerler]

Relations for Cob

[Kerler] necessary relations

I. General isotopies

- Artin Braid Relations
- a cross can be moved over Δ, μ, ν, ε

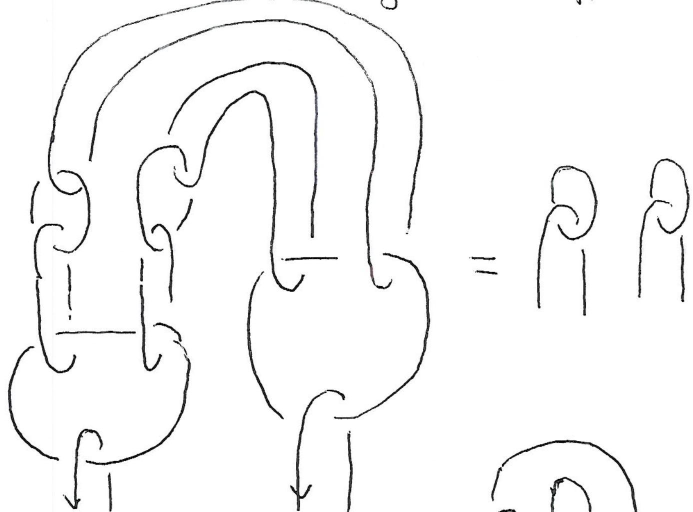
II. Axioms of Braided Hopf Algebras

(in our case $(1; \mu, \eta, \Delta, \varepsilon, S, S^{-1})$)

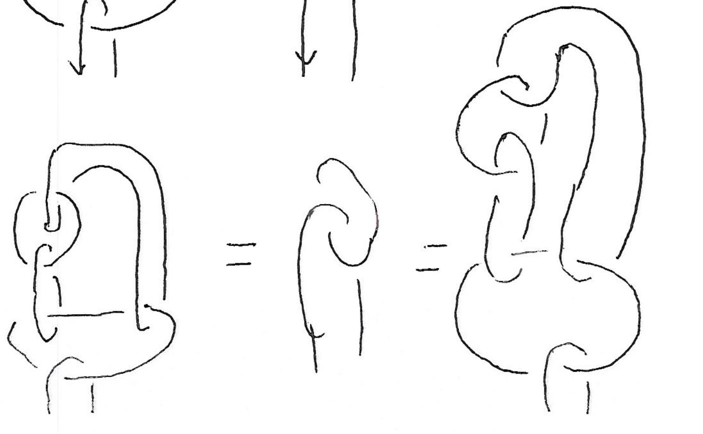
- associativity & units
- braided algebra
- relations for invertible antipode
- relations between ℓ, $S^{\pm 1}$, η, ε
- central & invertible Hopf pairings
- normalizations
- relations derived with convolutions

[Bobtcheva, Piergallini, 2006] two more relations give a sufficient set

- the duality of the algebra is integral and cointegral w.r.t. the copairing
- another normalization



Independently,
Habiro has proposed an alternative relation which completes Kerler's set to a sufficient one



2.2. The codomain of $\tilde{Z}$: category ${}^{ts}\mathcal{A}$

objects : non-negative integers

morphisms: elements of ${}^{ts}\mathcal{A}(g,f)$ $\stackrel{\text{def}}{=}$

$\mathbb{Q}$-vector space of top-substantial Jacobi diagrams with univalent vertices labeled by element of $\{1^+,\dots,g^+\}\cup\{1^-,\dots,f^-\}$

A Jacobi diagram is top-substantial if no component of the graph is a strut with both vertices colored by elem. of $\{1^+,\dots,g^+\}$

complete everything w.r.t. the degree of the diagr.

The composition :

$$\circ : {}^{ts}\mathcal{A}(g,f) \times {}^{ts}\mathcal{A}(h,g) \longrightarrow {}^{ts}\mathcal{A}(h,f)$$
$$x \qquad\qquad y \qquad\qquad x\circ y$$

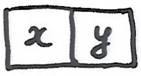
$x\circ y := x \sqcup y \,\Big/$ contract the i^+-colored vertices in x with the i^--colored vertices in y , $\forall i=\overline{1,g}$

The identity morphism of the object $g \in {}^{ts}\mathcal{A}$:

$$\mathrm{Id}_g = \exp_{\sqcup}\Big(\sum_{i=1}^{g} \Big|_{i^-}^{i^+}\Big)$$

Natural monoidal structure :

$x\circ y =$ 

$x\otimes y =$ 

Let $b \in {}^{ts}\mathcal{A}(h,g) = {}^{ts}\mathcal{A}\begin{pmatrix} h^+ \\ g^- \end{pmatrix}$

$a \in {}^{ts}\mathcal{A}(g,f) = {}^{ts}\mathcal{A}\begin{pmatrix} g^+ \\ f^- \end{pmatrix}$

$D \in \mathcal{M}_{h \times g}(\mathbb{Q})$

Define $D : \mathbb{Q}\cdot\{1^-, \dots, g^-\} \to \mathbb{Q}\cdot\overbrace{\{1^+, \dots, h^+\}}^{\lfloor h \rceil^+ \text{ notation}}$

$$D \cdot i^- = \sum_{j=1}^{h} d_{j^+ i^-} \cdot j^+$$

For a symmetric matrix A, denote $[A] := \exp\left(\sum_{i,j \in C} A_{ij} \Big|_i^j\right) \in \mathcal{A}(C)$ (C: a set)

> **Lemma.** Assume $a = \left[\frac{A}{2}\right] \sqcup a^Y$, $b = \left[\frac{B}{2}\right] \sqcup b^Y$, with A, B symmetric matrices of the form
>
> $$A = \begin{matrix} g^+\{ \\ f^-\{ \end{matrix}\begin{pmatrix} 0 & A^{+-} \\ \underbrace{A^{-+}}_{g^+} & \underbrace{A^{--}}_{f^-} \end{pmatrix} \qquad B = \begin{matrix} h^+\{ \\ g^-\{ \end{matrix}\begin{pmatrix} 0 & B^{+-} \\ \underbrace{B^{-+}}_{h^+} & \underbrace{B^{--}}_{g^-} \end{pmatrix}$$
>
> Then $a \circ b$ is also decomposible:
>
> $$a \circ b = \left[\frac{1}{2}\begin{pmatrix} 0 & B^{+-}A^{+-} \\ A^{-+}B^{-+} & A^{--} + A^{-+}B^{--}A^{+-} \end{pmatrix}\right] \sqcup \left(a^Y \overset{A,B}{\star} b^Y\right)$$
>
> where $\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-) \times \mathcal{A}^Y(\lfloor h \rceil^+ \cup \lfloor g \rceil^-) \xrightarrow{\cdot \overset{A,B}{\star} \cdot} \mathcal{A}^Y(\lfloor h \rceil^+ \cup \lfloor f \rceil^-)$
>
> $$x \overset{A,B}{\star} y := \left\langle x\big|_{i^+ \mapsto i^* + B^{+-}\cdot i^- + A^{-+}B^{--}\cdot i^-},\ \left[\frac{B^{--}}{2}\right]\Big|_{i^- \to i^*} \sqcup y\big|_{i^- \to i^* + A^{-+}\cdot i^+} \right\rangle_{\lfloor g \rceil^*}$$
>
> and $\langle\ ,\ \rangle_{\lfloor g \rceil^*}$ denotes contraction of i^*-vertices of left part with the i^*-vertices of right part, $\forall\, i = \overline{1,g}$
>
> Moreover, if a, b are grouplike, $a \circ b$ is also grouplike.

Examples:

1. $^{d}\mathcal{L}Cob$ $A = \begin{pmatrix} 0 & A^{+-} \\ A^{-+} & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & B^{+-} \\ B^{-+} & 0 \end{pmatrix}$

$$x \overset{A,B}{\star} y = \left\langle x\big|_{i^+ \mapsto i^* + B^{+-}\cdot i^-},\ y\big|_{i^- \mapsto i^* + A^{-+}\cdot i^+} \right\rangle_{\lfloor g \rceil^*}$$

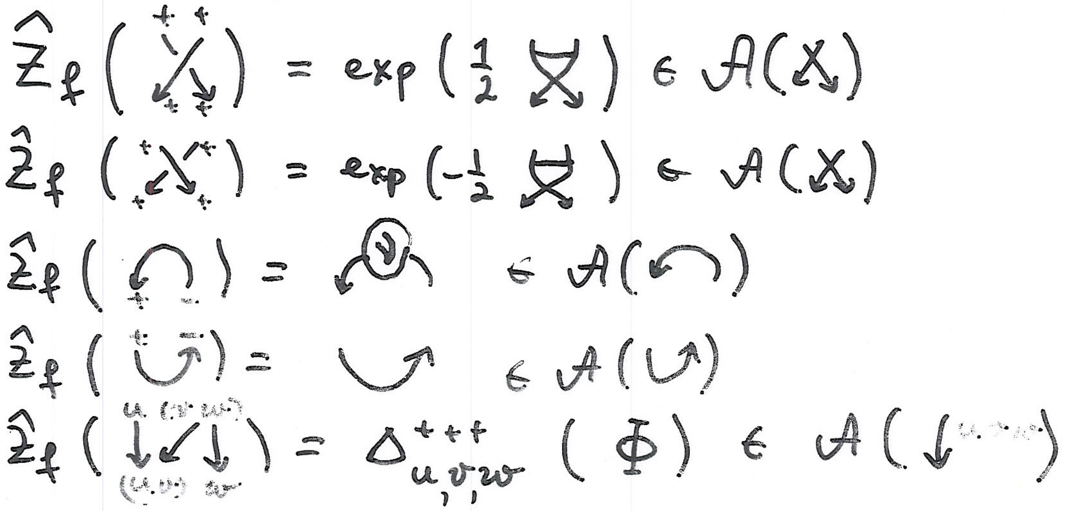
2. homology (Torelli) cylinders $A = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$$x \star y = \left\langle x\big|_{i^+ \to i^* + i^+},\ y\big|_{i^- \to i^* + i^-} \right\rangle_{\lfloor g \rceil^*}$$

2.3. Construction of $\tilde{Z} : \mathcal{L}Cob_q \to {}^{ts}\mathcal{A}$

$\hat{Z}_f : \mathcal{T}_q \longrightarrow \mathcal{A}$ — classical Kontsevich integral

($\mathcal{T}_q$: q-tangles; $\mathcal{A}$: Jacobi diagrams on abstract tangles)



$\hat{Z}_f(\ldots) = \exp\left(\frac{1}{2}\ldots\right) \in \mathcal{A}(\ldots)$

$\hat{Z}_f(\ldots) = \exp\left(-\frac{1}{2}\ldots\right) \in \mathcal{A}(\ldots)$

$\hat{Z}_f(\ldots) = \ldots \in \mathcal{A}(\ldots)$

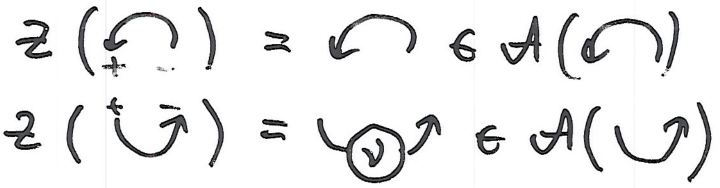
$\hat{Z}_f(\ldots) = \ldots \in \mathcal{A}(\ldots)$

$\hat{Z}_f(\ldots) = \Delta^{+++}_{u,v,w}(\Phi) \in \mathcal{A}(\ldots)$

$Z : \mathcal{T}_q \longrightarrow \mathcal{A}$ — modified Kontsevich integral

($\mathcal{T}_q$: q-tangles; $\mathcal{A}$: Jacobi diagrams on abstract tangles)

$$Z(\gamma) := \hat{Z}_f(\gamma) \#_1 \nu^{d(\gamma_1)} \#_2 \nu^{d(\gamma_2)} \# \ldots \#_\ell \nu^{d(\gamma_\ell)}$$



$Z(\ldots) = \ldots \in \mathcal{A}(\ldots)$

$Z(\ldots) = \ldots \in \mathcal{A}(\ldots)$

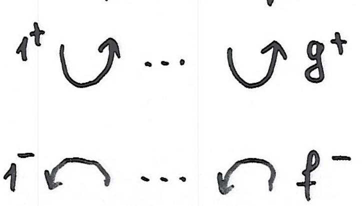
$$d(\gamma_j) = \begin{cases} -1 & \text{bottom-bottom} \\ 0 & \text{bottom-top, or} \\ 1 & \text{top-top} \end{cases}$$

$Z : \mathcal{T}_q Cub \longrightarrow \mathcal{A}$ — typical Kontsevich-LMO invariant

q-tangles in homology cubes $\ni (B, \gamma)$

Jacobi diagrams on strands (1-manifolds)

for example on



$1^+ \ldots g^+$

$1^- \ldots f^-$

(B, γ) = q-tangle in a homology cube

$\downarrow$

$([-1,1]^3, \gamma)_L$ = surgery presentation, $L \subset [-1,1]^3$ a link

$$Z(B,\gamma) := U_+^{-\sigma_+(L)} \sqcup U_-^{-\sigma_-(L)} \sqcup \int_{\pi_0(L)} \chi^{-1}_{\pi_0(L)} Z(L^\nu \cup \gamma) \in \mathcal{A}(\gamma$$

where

- $$Z(L^\nu \cup \gamma) := Z(L \cup \gamma) \#_{\pi_0(L)} \nu^{\otimes \pi_0(L)} \in \mathcal{A}(L \cup \gamma)$$
- $(\sigma_+(L), \sigma_-(L))$ = signature of the linking matrix
- $$U_\pm := \int \chi^{-1} (\underbrace{\nu \# Z(\bigcirc^{\pm 1})}_{= Z(\bigcirc^\nu_{\pm 1})}) \in \mathcal{A}(\emptyset)$$
- $\chi = \chi_S : \mathcal{A}(X, C \cup S) \to \mathcal{A}(X \cup S, C)$

  $\chi_S(D) \equiv$ average of all possible ways of attaching, for all $s \in S$, s-colored external vertices of D to s-indexed interval / circle
- Formal Gaussian integral of $G \in \mathcal{A}(X, C \cup S)$ along S

  $$G = \left[\frac{L}{2}\right] \sqcup P$$

  L = symmetric matrix (gives the strut part), $\det L \neq 0$

  P = S-substantial (does not contain struts with both vertices in S)

  $$\int_S G := \left\langle \left[-\frac{L^{-1}}{2}\right], P \right\rangle_S \in \mathcal{A}(X, C)$$

  contraction along S

Lemma $\left\{ \begin{array}{l} \chi^{-1} Z(B,\gamma) \in \mathcal{A}(\pi_0(\gamma)) \\ \text{is group-like and} \\ \text{has s-reduction } \left[\frac{lk_B(\gamma)}{2}\right] \end{array} \right.$

Symmetrized Kontsevich-LMO invariant

$$\chi^{-1} Z : \mathcal{L}Cob_q \longrightarrow {}^{ts}\mathcal{A}$$

$$\chi^{-1} Z : \begin{array}{c} \overset{g}{M} \\ f \\ \rotatebox{90}{$\in$} \\ \mathcal{L}Cob_q(w,v) \end{array} \longmapsto \begin{array}{c} \mathcal{A}(\lfloor g \rceil^+ \cup \lfloor f \rceil^-) \\ \| \\ {}^{ts}\mathcal{A}(|w|,|v|) \end{array}$$

$M \rightsquigarrow$ represented by a bottom-top tangle (B,γ)

Lemma. M Lagrangian q-cobordism $\Rightarrow \chi^{-1}Z(M)$ top-substantial

---

$$\boxed{\tilde{Z}(M) := \chi^{-1} Z(M) \circ T_g} \in \mathcal{A}(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$$

normalized Kontsevich-LMO invariant

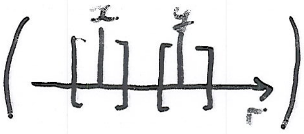
Main lemma. M, N Lagrangian q-cobordisms such that $M \circ N$ makes sense. Then:

$$\boxed{\chi^{-1} Z(M \circ N) = \chi^{-1} Z(M) \circ T_g \circ \chi^{-1} Z(N)}$$

Lemma. M Lagrangian q-cobordism $\Rightarrow$
$\Rightarrow$ $\tilde{Z}(M)$ is group-like and has s-reduction $\left[\frac{lk(M)}{2}\right]$.

$T_g \in \mathcal{A}(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$

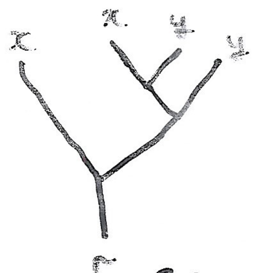
x, y, r = formal variables

$\lambda(x,y,r) = x^{-1}\Big($  $\Big) \in \mathcal{A}(\{x,y,r)$

It can be computed using Baker–Campbell–Hausdorff series:

$$\log(\exp(x)\cdot\exp(y)) \in \mathbb{Q}[[x,y]]$$

Writing Lie commutators as r-rooted binary trees with leaves colored by x and y:

e.g. $[x,[[x,y],y]] \longmapsto$ 

defines an embedding of $\mathrm{Lie}(x,y)$ into $\mathcal{A}^c(\{x,y,r\})$

Hence

$$\lambda(x,y,r) = \exp(\underbrace{\Lambda(x,y,r)}_{\text{defined by the BCH series}})$$

$$\overbrace{\overset{x}{\underset{r}{\mid}} + \overset{y}{\underset{r}{\mid}} + \Lambda^Y(x,y,r)} \in \mathcal{A}^c(\{x,y,r\})$$

Set $T_g = T(1^+,1^-) \sqcup \ldots \sqcup T(g^+,g^-)$,

where $T(x_+,x_-) = U_+^{-1}U_-^{-1}\int_r \langle \lambda(x_-,y_-,r_-) \sqcup \lambda(x_+,y_+,r_+),$

$, x^{-1}Z\Big($  $\Big)\rangle_y \in \mathcal{A}(\{x_+,x_-\})$

$T(x_+, x_-)$ and T_g — group like, top-substantial, has s-reduction $\exp\Big($  $\Big) = Id_g$

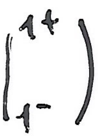
Proof of main lemma uses functoriality (for q-tangles) of the Kontsevich integral, the formula

$$\sigma_\pm(K \cup J_g \cup L) = \sigma_\pm(K) + g + \sigma_\pm(L)$$

and that $T_w = T_g$ (independent of non-associative word).

Proof of functoriality theorem uses the above two lemmas. (One needs also to check that $\tilde{Z}^Y(Id_w) = 1$).

For an even Drinfeld associator:

$T_1 = \exp\Big($  $\Big) \sqcup \Big(1 - \frac{1}{8}$  $- \frac{1}{48}$  $+$

$+ \frac{1}{8}$  $+ (i\text{-deg} > 2)\Big) \in {}^{ts}\mathcal{A}(1,1)$

3. Properties and applications

3.1. Reduction to a functor $\widetilde{Z}: \widehat{\mathcal{L}Cob_q} \to {}^{ts}\hat{A}$ from the category of Lagrangian cobordisms between closed surfaces

$\widehat{F_g} = F_g \cup D$

$\widehat{C_{g-}^{g+}} = C_{g-}^{g+} \cup (D \times [-1,1])$

$Cob \to \widehat{Cob}$

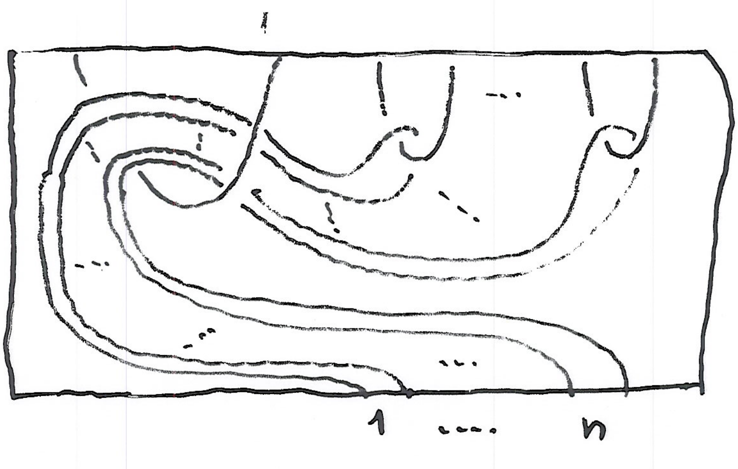
$\mathcal{L}Cob \to \widehat{\mathcal{L}Cob}$

$\mathcal{L}Cob_q \to \widehat{\mathcal{L}Cob_q}$

} glue a 2-handle along the "vertical" boundary of cobordisms

Same objects, surjections on morphisms

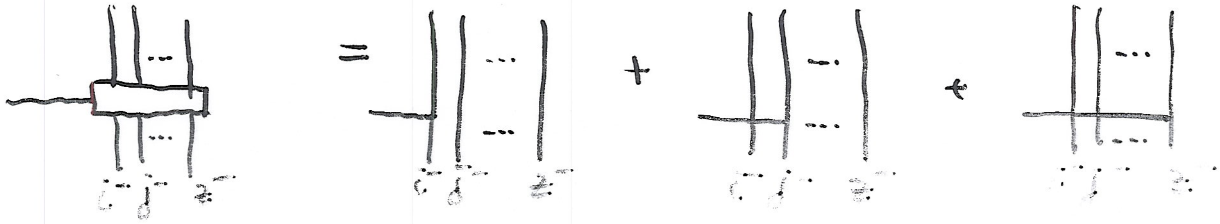
Proposition. The functor $\hat{\cdot}: Cob \to \widehat{Cob}$ induces an isomorphism $\hat{\cdot}: Cob/\sim \to \widehat{Cob}$, where $\sim$ is the relation generated by $\alpha_n \sim \varepsilon \otimes Id_n$, $\forall n \geq 0$.

$\alpha_n =$  $\in {}^s\mathcal{L}Cob(n+1, n)$

Similar statements hold for $\mathcal{L}Cob$ and ${}^s\mathcal{L}Cob$.

$$ {}^{ts}\hat{\mathcal{A}}(g,f) := {}^{ts}\mathcal{A}(g,f) \big/ \mathcal{I}(g,f) \quad , \forall f,g \geq 0 $$

where $\mathcal{I}(g,f)$ = the closure of the subspace of ${}^{ts}\mathcal{A}(g,f)$ spanned by Jacobi diagrams D that contain



where the drawn part of D contains <u>all</u> external vertices of D labeled by elements of $\{1^-, \dots, f^-\}$.

Theorem. The functor $\tilde{Z}$ reduced modulo $\mathcal{I}$ factors through $\widehat{\mathcal{L}Cob_q}$:

$$
\begin{array}{ccc}
\mathcal{L}Cob_q & \xrightarrow{\ \tilde{Z}\ } & {}^{ts}\mathcal{A} \\
\hat{\cdot} \downarrow & & \downarrow \hat{\cdot} \\
\widehat{\mathcal{L}Cob_q} & \overset{\tilde{Z}}{\dashrightarrow} & {}^{ts}\hat{\mathcal{A}}
\end{array}
$$

The contravariant "Hom-dual" of $\tilde{Z}$, ${}^{ts}\mathcal{A}(\tilde{Z}(\cdot), \overset{\sigma_j}{0})$ is, when restricted to doubly Lagrangian cobordisms, essentially the C-Le functor:

$$
\begin{array}{ccccccc}
\mathcal{A}(\lfloor f \rceil^{+}) & \xleftarrow[\cong]{\langle \cdot , T_f \rangle} & \mathcal{A}(\lfloor f \rceil^{+}) & \xrightarrow[\cong]{\chi} & \mathcal{A}(\circlearrowleft^{\lfloor f \rceil}) & \xrightarrow[\cong]{\#(\nu^{1/2})^{\otimes \lfloor f \rceil}} & \mathcal{A}(\circlearrowleft^{\lfloor f \rceil}) \\
\Big\downarrow {}^{ts}\mathcal{A}[\tilde{Z}(\hat{M}), \sigma] & & & & & & \Big\downarrow \tau(\hat{M}) \\
\mathcal{A}(\lfloor g \rceil^{+}) & \xleftarrow[\cong]{\langle \cdot , T_g \rangle} & \mathcal{A}(\lfloor g \rceil^{+}) & \xrightarrow[\cong]{\chi} & \mathcal{A}(\circlearrowleft^{\lfloor g \rceil}) & \xrightarrow[\cong]{\#(\nu^{1/2})^{\otimes \lfloor g \rceil}} & \mathcal{A}(\circlearrowleft^{\lfloor g \rceil})
\end{array}
$$

Advantage: Do not need to extend the construction of τ to embedded chain graphs .

Corollary. $\tau(\hat{M})$ is independent of the Drinfeld associator Φ chosen.

3.2. Universality of $\tilde{Z}^Y$ for FTI of Lagrangian cobordisms

Fix $\mathcal{M}^0$ a Y_1-equivalence class of 3-manifolds

$$\mathbb{Q}\cdot\mathcal{M}^0 \supseteq \mathcal{F}_1(\mathcal{M}^0) \supseteq \mathcal{F}_2(\mathcal{M}^0) \supseteq \mathcal{F}_3(\mathcal{M}^0) \supseteq \dots$$

$$\mathcal{F}_n(\mathcal{M}^0) = \Big\langle \underbrace{\sum_{G' \subset G} (-1)^{|G'|} M_{G'}}_{=:\,[M,G]} \;\Big|\; \begin{matrix} M \in \mathcal{M}^0 \\ G \text{ graph clasper in } M \\ \text{of i-deg } k \end{matrix} \Big\rangle$$

$G' \subset G$: subset of the set of connected components

$[M, G]$ surgery bracket along G

$$\mathcal{G}(\mathcal{M}^0) = \prod_{i \geq 0} \mathcal{G}_i(\mathcal{M}^0) := \prod_{i \geq 0} \frac{\mathcal{F}_i(\mathcal{M}^0)}{\mathcal{F}_{i+1}(\mathcal{M}^0)}$$

Theorem [Garoufalidis] Let $\mathcal{M}^0$ be a Y_1-equiv. class of Lagrangian cobordisms from F_g to F_f. Then there exists a surjective graded map

$$\sigma : \mathcal{A}(\lfloor g \rceil^+ \cup \lfloor f \rceil^-) \longrightarrow \mathcal{G}(\mathcal{M}^0)$$

which realizes each Jacobi diagram as a graph clasper in a representative of $\mathcal{M}^0$, and performs surgery along it:

$$\sigma_i(D) := [M, \underbrace{C(D)}_{\text{topological realization of } D}] \bmod \mathcal{F}_{i+1}(\mathcal{M}).$$

$C(D)$: at the external $+$ vertices : take a push-off into M° of $m_-^\circ(\alpha_j) \subset \partial M^\circ$

: at the external$-$ vertices : take a push-off into M° of $m_+^\circ(\beta_j) \subset \partial M^\circ$

Theorem (universality) Let $\mathcal{M}^\circ$ be a Y_1-equivalence class of Lagrangian cobordisms from F_g to F_f. Then, the i-degree i part of

$$\tilde{Z}^Y : \mathcal{M}^\circ \longrightarrow \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$$

$$\text{words} \rightsquigarrow w_t(M),\ w_b(M) \quad M \longmapsto \tilde{Z}^Y$$

is a finite-type invariant of degree i. Moreover, the induced map

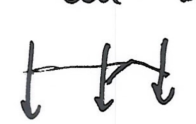
$$Gr\,\tilde{Z}^Y : \mathcal{G}(\mathcal{M}^\circ) \to \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$$

gives (up to a sign) a right-inverse to the surgery map σ. In particular, both are isomorphis[ms]

$$(Gr\,\tilde{Z}^Y) \circ \sigma(D) = (-1)^{\text{internal deg} + \#\text{conn. comp} + \#\text{internal edges}} \cdot D$$

(internal deg, # conn. comp, # internal edges: of D)

Proof uses Φ an even associator

$\Phi = 1 + \frac{1}{24}$  $+ (\deg \geq 3)$

Corollary. $g, f \geq 0$. $\forall D \in \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor f \rceil^-)$ of i-deg i, $\forall$ non-assoc. words v, w of length f, resp. there exists $M \in \mathcal{L}Cob_q(w, v)$ s.t.

$$\tilde{Z}^Y(M) = 1 + D + (i\text{-deg} > i)$$

3.3. Finite-dimensional representations of the Torelli group

$$i\text{-}\deg_{\leq n} \tilde{Z}^Y : M \longmapsto \underset{\text{element}}{\overset{\text{grouplike}}{}} \in i\text{-}\deg_{\leq n} {}^{ts}\mathcal{A}$$

$${}^{ts}\mathcal{A}(g,g) \times {}^{ts}\mathcal{A}(0,g) \longrightarrow {}^{ts}\mathcal{A}(0,g)$$

$f \in MCG_{g,1}$

$$M_f = \left(F_g \times [-1,1] \;,\; Id \times (-1) \cup f \times 1 \right)$$

Example. $i\text{-}\deg_{\leq 2} \tilde{Z}^Y : \mathcal{J}_{g,1} \to GL(d, \mathbb{Q})$

$$d = \frac{1}{72}(g+1)\left(g^5 - 7g^4 + 26g^3 - 20g^2 + 72\right)$$

g	d
0	1 ← Id
1	2 ← Id
2	5
3	15
4	45
5	131
6	357
7	890
8	2025
9	4240
10	8261

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

3.4. Applications to homology (Torelli) cylinders

$$\mathcal{I}_{g,1} \cap \mathrm{Cyl}(F_g) = \{ (M,m) \in \mathcal{C}ob(g,g) \mid m_{\pm *}: H_1(F_g) \to H_1(M) \text{ are isom., and } m_{+*} = m_{-*} \} \cap \mathcal{L}\mathcal{C}ob(g,g)$$

$$\mathrm{lk}(M) = \begin{pmatrix} 0 & I_g^{+-} \\ I_g^{-+} & 0 \end{pmatrix}$$

Corollary (functoriality) The renormalized LMO invariant of homology cylinders defines a monoid homomorphism

$$\tilde{Z}^Y : (\mathrm{Cyl}(F_g), \circ) \to (\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-), \star)$$

where

$$D \star E = \left\langle D|_{i^+ \to i^+ + i^*}, E|_{i^- \to i^- + i^*} \right\rangle_{\lfloor g \rceil^*}$$

Corollary (universality) The algebra homomorphism $\tilde{Z}^Y : \mathbb{Q}[\mathrm{Cyl}(F_g)] \to \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-)$ sends the Y-filtration to the i-deg. filtration and induces isomorphisms at the graded level. Moreover, $\mathrm{Gr}\,\tilde{Z}^Y : \mathcal{G}(\mathrm{Cyl}(F_g)) \to \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-)$ is, up to explicite sign, the inverse of the surgery map σ

[Garoufalidis, Levine, 2005] σ surjective, sends $\star$ to $\circ$.

[Habegger, 2000] domain & codomain of $\mathrm{Gr}\,\tilde{Z}^Y$ are iso as vect. sp.

Graded Lie algebra of homology (Torelli) cylinders

$Cyl_k(F_g) := \{ M \in Cyl(F_g) \,/\, M \text{ is } Y_k\text{-equivalent to } F_g \times [-1,1] \}$

Gusarov, Habiro: $Cyl_k(F_g)/Y_\ell$ is a group for $\ell \geq k \geq 1$
and $\left[Cyl_k(F_g)/Y_\ell \,,\, Cyl_{k'}(F_g)/Y_\ell \right] \subset Cyl_{k+k'}(F_g)/Y_\ell$
for $\ell \geq k+k'$, $k, k' \geq 1$

Denote $\overline{Cyl}(F_g) := \prod_{i \geq 1} \frac{Cyl_i(F_g)}{Y_{i+1}} \otimes \mathbb{Q}$

On the diagramatic side, look at the primitives in $\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-)$:

$\mathcal{A}^{Y,c}(\lfloor g \rceil^+ \cup \lfloor g \rceil^-) :=$ subspace spanned by nonempty connected Jacobi diagrams

$[x, y] := x \star y - y \star x$

Theorem. The LMO homomorphism of homology (Torelli) cylinders induces a Lie algebra iso:

$$Gr\, \tilde{Z}^Y : \overline{Cyl}(F_g) \longrightarrow \mathcal{A}^{Y,c}(\lfloor g \rceil^+ \cup \lfloor g \rceil^-)$$

that, for all $M \in Cyl_i(F_g)$ sents $\{M\} \otimes 1$ to the i-deg i part of $\tilde{Z}^Y(M)$.

($\otimes$ above $\otimes_{\mathbb{Z}}$)

$\downarrow \varphi$

$\mathcal{A}^c(F_g)$

$\tilde{Z}^Y$ depends on the choice of symplectic basis (α, β)

$\varphi \circ Gr\, \tilde{Z}^Y$ does not