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Magnus representations of the mapping class group and L^2-torsion invariants

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Joint work with

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  - Proc. Japan Academy, Vol. 79, ser. A. No. 4, 2003.
  - J. Math. Soc. Japan Vol. 56, No. 2, 2004.
- T. Morifuji:
  - arXiv:0801.4429(math.GT).
  - In progress.

1 Plan of my talk

The main subjects of my talk;

- Magnus representation,
- L^2-torsion.

I want to explain mainly L^2-torsion, and in particular Fuglede-Kadison determinant which is the main tool to define it.

Plan

1. Determinant in Linear Algebra
2. Fuglede-Kadison determinant
3. Magnus representation of the mapping class group
4. L^2-torsion
5. Nilpotent quotient and L^2-torsion invariants
6. Results

2 Determinant in Linear Algebra

For a matrix $B \in M(n;\mathbb{C})$,

- $\mathrm{tr}(B)$, $\det(B)$, or more generally, symmetric polynomials of the eigenvalues,
- the characteristic polynomial $\det(tE - B)$,

are fundamental quantities of B. Here

- E: the identity matrix,
- t: the variable of the characteristic polynomial.

We want to define a kind of

determinant over non-commutative rings,

which is group rings of fundamental groups in mind.

Determinant

Recall one of the definitions of the determinant. Not standard, but well known in the are of zeta function theory, dynamical systems, or spectral geometry.

fundamental equality:

$$\log|\det(B)|\text{''} = \text{''}\mathrm{tr}(\log(B)).$$

We want to explain more precisely the above. It can be generalized over the group algebra.

Most simple case: A diagonal matrix.

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & & \dots & & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Here, we assume that the eigenvalues are

$$0 < \lambda_1, \dots, \lambda_n < 1.$$

Directly we compute,

$$\begin{aligned}\log(\det(B)) &= \log(\lambda_1 \cdots \lambda_n) \\ &= \sum_{i=0}^{n} \log(\lambda_i) \\ &= \sum_{i=0}^{n} \log(1 + (\lambda_i - 1)).\end{aligned}$$

Here recall the expansion of $\log(1+x)$ at $x = 0$

$$\log(1+x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p.$$

Then

$$
\begin{aligned}
\log(\det(B)) &= \sum_{i=0}^{n}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p}(\lambda_i - 1)^p\right)\\
&= -\sum_{i=0}^{n}\left(\sum_{p=1}^{\infty}\frac{1}{p}(1-\lambda_i)^p\right)\\
&= -\sum_{p=1}^{\infty}\frac{1}{p}\left(\sum_{i=0}^{n}(1-\lambda_i)^p\right).
\end{aligned}
$$

Hence, we can get the following equality:

$$\det(B) = \exp\left(-\sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}\left((E-B)^p\right)\right).$$

or equivalently,

$$\log\det(B) = -\sum_{p=1}^{\infty} \frac{1}{p} \operatorname{tr}\left((E-B)^p\right).$$

General case :

- Non diagonal matrix case :
  - Symmetric matrix, or Hermitian matrix.
    - replace B to BB^*(B^*:the adjoint matrix of B).
    - For BB^*, eigenvalue is changed from λ_i of B to $\lambda_i \bar{\lambda_i} = |\lambda_i|^2$ of BB^*.

- some engenvalue $|\lambda_i| > 1$:
  The problem is that the convergence radius of $\log(1+x)$ equals 1.
  - For a sufficiently large constant $K > 0$ such that $0 < \lambda/K < 1$,

$$\log(\lambda) = \log\left(K\frac{\lambda}{K}\right) = \log(K) + \log\left(\frac{\lambda}{K}\right).$$

– replace B to $\dfrac{1}{K}B$ (equivalently, BB^* to $\dfrac{1}{K^2}BB^*$).

Summary:

For any matrix $B \in GL(n;\mathbb{C})$,

$$|\det(B)| = K^{2n} \exp\left(-\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{tr}(E - \frac{1}{K^2}BB^*)^p \right).$$

We extend this equality to the one in the non commutative group algebra as the definition of $|\det(B)|$.
Our targets are group rings of fundamental group of 3-manifolds, or 2-manifolds.

3 Fuglede-Kadison determinant

Origin: Theory of the von Neumann algebra.

- Fuglede-Kadison:Determinant theory in finite factor, Ann. of Math. (2), **55** (1952).

In this talk, we treat only group (von Neumann) algebra cases.

Why we need the operator theory ?

One reason is that $\mathbb{C}\pi$ is not a Noetherian ring.
It means,
for finitely generated $\mathbb{C}\pi$-module C and its submodule D, it is not guaranteed that its quotient module C/D is finitely generated, in general.
It is obstruction to handle directly the homology, or cohomology theory over the group ring.

Here we fix some notations:

- π: a group.
- e: the unit of π.
- $\mathbb{C}\pi$: the group algebra of π over $\mathbb{C}$(a linear space over $\mathbb{C}$).
- $l^2(\pi)$: l^2-completion of $\mathbb{C}\pi$, namely, algebra of all infinite sums $\sum_{g\in\pi}\lambda_g g$ such that $\sum_{g\in\pi}|\lambda_g|^2 < \infty$.

By using the equality $\log|\det| = \operatorname{tr}\log$, if we can define tr, we can do $\det$.
First the trace over $\mathbb{C}\pi$ is defined as follows.

Definition 3.1 $\mathbb{C}\pi$-trace:

$$\operatorname{tr}_{\mathbb{C}\pi}\left(\sum_{g\in\pi}\lambda_g g\right) = \lambda_e \in \mathbb{C}.$$

This $\mathbb{C}\pi$-trace $\operatorname{tr}_{\mathbb{C}\pi} : \mathbb{C}\pi \to \mathbb{C}$ can be naturally extended to the trace on the matrices over $\mathbb{C}\pi$.

For a matrix $B = (b_{ij}) \in M(n; \mathbb{C}\pi)$,

$$\mathrm{tr}_{\mathbb{C}\pi}(B) = \sum_{i=1}^{n} \mathrm{tr}_{\mathbb{C}\pi}(b_{ii}).$$

By using this trace

$$\mathrm{tr}_{\mathbb{C}\pi} : M(n; \mathbb{C}\pi) \to \mathbb{C},$$

Fuglede-Kadison determinant is defined as follows.

Definition 3.2 Fuglede-Kadison determinant:

$$\det_{\mathbb{C}\pi}(B) = K^{2n} \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{tr}_{\mathbb{C}\pi}\left(E - \frac{BB^*}{K^2}\right)^p\right) \in \mathbb{R}_{>0}.$$

Here

- $K > 0$: a sufficiently large constant.
- $B^* = (\overline{b_{ji}})$: the adjoint matrix of $B = (b_{ij})$.

The adjoint matrix B^* is defined by

- the complex conjugate of coefficients,
- antihomomorphism :

$$\overline{\sum \lambda_g g} := \sum \overline{\lambda}_g g^{-1}.$$

Remark 3.3 The matrix B can be consider the operator on Hilbert space $l^2(\pi)^n$, and then the above adjoint matrix is just the adjoint operator in the usual sense.

Remark 3.4 The convergence of the infinite series is not trivial. However, it is known that under some general condition of the group π,

- If L^2-betti number of B

$$\lim_{p\to\infty}\left(\frac{1}{p}\mathrm{tr}_{\mathbb{C}\pi}\left(\left(E-K^{-2}BB^*\right)^p\right)\right)=0,$$

then it is guaranteed.

Example 3.5 In the case of

- a free group of a finite rank,
- a nilpotent group,
- an amenable group,
- a hyperbolic group,

the Fugkede-Kadioson determinant converges if L^2-betti number is vanishing.

4 Magnus representation

- $\Sigma_{g,1}$: oriented compact surface of a genus $g \geq 1$ with 1 boundary component.

- $* \in \partial\Sigma_{g,1}$: a base point of $\Sigma_{g,1}$.

- $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1}))$: the mapping class group of $\Sigma_{g,1}$.

- $\Gamma = \pi_1(\Sigma_{g,1}, *)$: free group of rank $2g$.

- $\langle x_1, \ldots, x_{2g} \rangle$: a generating system of Γ.

- $\varphi_* \in Aut(\Gamma)$: the induced automorphism by $\varphi \in \mathcal{M}_{g,1}$.

Proposition 4.1 (Dehn-Nielsen-Zieschang)

$$\mathcal{M}_{g,1} \ni \varphi \mapsto \varphi_* \in Aut(\Gamma)$$

is injection.

Under fixing generator $\{x_1, \ldots, x_{2g}\}$, a mapping class φ can be determined by the words

$$\varphi_*(x_1), \ldots, \varphi_*(x_{2g}).$$

The Magnus representation of the mapping class group is defined as follows.

Definition 4.2 Magnus representation:

$$r : \mathcal{M}_{g,1} \ni \varphi \mapsto \left(\overline{\frac{\partial \varphi_*(x_j)}{\partial x_i}} \right)_{i,j} \in GL(2g; \mathbb{Z}\Gamma).$$

Here

- $\partial/\partial x_1, \ldots, \partial/\partial x_{2g} : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ are the Fox's free differentials.

- The conjugation on $\mathbb{Z}\Gamma$ is defined as follows. For any element $\sum_g \lambda_g g \in \mathbb{Z}\Gamma$,

$$\overline{\sum_g \lambda_g g} = \sum_g \lambda_g g^{-1}.$$

Recall Fox's free differentials

- $\dfrac{\partial x_j}{\partial x_i} = \delta_{ij}$,
- $\dfrac{\partial}{\partial x_i}(\gamma\gamma') = \dfrac{\partial \gamma}{\partial x_i} + \gamma \dfrac{\partial \gamma'}{\partial x_i}$ $(\gamma, \gamma' \in \pi)$,
- it is extended as a $\mathbb{Z}$-linear map.

Remark 4.3 This map is not a homomorphism, but a crossed homomorphism. According to the practice, it is called the Magnus representation of $\mathcal{M}_{g,1}$.

By taking the abelianization

$$\Gamma = \pi_1(\Sigma_{g,1}) \to H = H_1(\Sigma_{g,1};\mathbb{Z}),$$

the map $r_2 : \mathcal{M}_{g,1} \to GL(2g;\mathbb{Z}H)$ is obtained. If we restrict this map to the Torelli group

$$\mathcal{I}_{g,1} = \mathrm{Ker}\{\mathcal{M}_{g,1} \to \mathrm{Sp}(2g;\mathbb{Z})\},$$

$$r_2 : \mathcal{I}_{g,1} \to GL(2g;\mathbb{Z}H)$$

is a homomorphism.

5 L^2-torsion invariants

The characteristic polynomial of the image of the Magnus representation $r(\varphi) \in GL(n; \mathbb{C}\Gamma)$ can be considered as the Fuglede-Kadison determinant of $tE - r(\varphi)$. Final problem is ;

How can we consider the variable t?

For the mapping class $\varphi \in \mathcal{M}_{g,1}$, we take its mapping torus

$$W_\varphi := \Sigma_{g,1} \times [0,1]/(x,1) \sim (\varphi(x),0).$$

From here, we put

$$\pi = \pi_1(W_\varphi, *).$$

We fix a base point

$$* \in \partial\Sigma_{g,1} \times \{0\} \subset \Sigma_{g,1} \times \{0\} \subset W_\varphi.$$

Now the group π has the following presentation:

$$\pi = \left\langle x_1, \cdots, x_{2g}, t \mid r_1, \ldots, r_n \right\rangle,$$

where $r_i := tx_it^{-1}\left(\varphi_*(x_i)\right)^{-1} (i = 1...2g)$ and t is the generator of $\pi_1 S^1 \cong \mathbb{Z}$.

We can consider the variable "t" of the characteristic polynomial as the S^1-direction element in the fundamental group.

Put together,
in the $\mathbb{C}\pi \cong \mathbb{C}(\Gamma \rtimes \mathbb{Z})$, we can consider the characteristic polynomial, as a real number, of the image of the Magnus representation by using the Fuglede-Kadison determinant.

What is the geometric meaning ?
By the theorem of Lück,

$$-2\log\det_{\mathbb{C}\pi}(tE - r(\varphi))$$

is the L^2-torsion of the 3-manifold W_φ for the regular representation of π.

Remark 5.1 • L^2-torsion [Lott, Lück, Carey, Mathai, ...] is a generalization of Reidemeister-Ray-Singer torsion to the torsion invariant with infinite unitary representation.

- Recall that the natural linear space with actions of the group π is $\mathbb{C}\pi$, and its natural completion is $l^2(\pi)$. It is the regular representation of π.

Let us denote $\rho(\varphi)$ by the L^2-torsion of W_φ.

More precisely, we see the Lück's formula.
Applying the Fox free differentials to the relators $r_1, \cdots, r_{2g}$ of π, we obtain Fox matrix

$$A := \left(\frac{\partial r_i}{\partial x_j} \right)_{i,j} \in M(2g; \mathbb{Z}\pi).$$

Theorem 5.2 (Lück)

$$\log \rho(\varphi) = -2 \log \det_{\mathbb{C}\pi}(A).$$

By the definition,

$$A = tE - \overline{{}^{t}r(\varphi)}.$$

It is easy to see

$$\det_{\mathbb{C}\pi}(A) = \det_{\mathbb{C}\pi}(tE - r(\varphi)).$$

Again we want to ask what is the geometric meaning of L^2-torsion :

Answer:it is the hyperbolic volume!!

Theorem 5.3 (Lott, Schick,...) For any hyperbolic 3-manifold M,

$$\log \rho(M) = -\frac{1}{3\pi} vol(M).$$

Remark 5.4 We can say that theoretically Lück's formula gives the way to compute the volume of the mapping torus W_φ from the actions of φ on the fundamental group.

6 Series of L^2-torsion invariants

We want to find more computable invariant.

Fundamental framework:

Lower central series and nilpotent quotients.

Lower central series of Γ:

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots$$

- $\Gamma_1 = \Gamma$,
- $\Gamma_k := [\Gamma_{k-1}, \Gamma_1]$ $(k \geq 2)$.
- $N_k := \Gamma/\Gamma_k$:k-th nilpotent quotient

- $p_k : \Gamma \longrightarrow N_k$: natural projection.

We obtain the series of (graded) Magnus representations

$$r_k : \mathcal{M}_{g,1} \longrightarrow GL(2g; \mathbb{Z}N_k).$$

Because $\Gamma_k \lhd \Gamma$, then

$$\pi(k) = \pi/\Gamma_k \cong N_k \rtimes \mathbb{Z}.$$

Then

$$p_k : \pi \longrightarrow \pi(k) = N_k \rtimes \mathbb{Z},$$

and we write $p_{k*} : \mathbb{C}\pi \longrightarrow \mathbb{C}\pi(k)$ to its induced homomorphism on the group ring.

k-th Fox matrix

$$A_k := \left(p_{k*} \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M(2g; \mathbb{C}\pi(k)).$$

[**Morifuji-Takasawa-Kitano**].
For W_φ, k-th L^2-torsion invariant $\rho_k(\varphi)$ can be defined and **,**

$$\log \rho_k(\varphi) = -2 \log \mathrm{det}_{\mathbb{C}\pi(k)}(A_k).$$

Remark 6.1 k-th L^2-torsion invariant $\rho_k(\varphi)$ can be counted the characteristic polynomial of $r_k(\varphi) \in GL(2g; \mathbb{C}\pi(k))$ in terms of Fuglede-Kadison determinant.

7 Results

By the general theory of torsion invariants, we can see,

Proposition 7.1

$$\rho_k(\varphi^n) = \rho_k(\varphi)^n$$

By taking $\log$, we obtain,

$$\log \rho_k(\varphi^n) = n \log \rho_k(\varphi).$$

Because L^2-torsion is a topological invariant of mapping torus, then we can see the following.

Proposition 7.2 For any mapping class $\varphi \in \mathrm{Ker}\{\mathcal{M}_{g,1} \to \mathcal{M}_g\}$,

$$\rho_k(\varphi) = 1.$$

Remark 7.3 For φ as above, topologically $W_\varphi \cong \Sigma_{g,1} \times S^1$.

Recall Nielsen-Thurston classification of the mapping classes:

- periodic
- reducible
- pseudo Anosov

In $\mathcal{M}_{g,1}$, there is no periodic element. However, for a lift of a periodic element in $\mathcal{M}_g$, i.e., for any mapping class $\varphi \in \mathcal{M}_{g,1}$ such that $\varphi^n \in \mathrm{Ker}\{\mathcal{M}_{g,1} \to \mathcal{M}_g\}$, we can see the following.

Proposition 7.4 $\rho_k(\varphi) = 1$ for any φ as above.

Proof. Remember that ρ_k is a positive real number.
For such φ as above,

$$\rho_k(\varphi^n) = 1.$$

On the other hand,

$$\rho_k(\varphi^n) = \rho_k(\varphi)^n.$$

Proposition 7.5 For $\varphi \in \mathcal{M}_{g,1}$, if there exists a separating simple closed curve $\gamma \subset \Sigma_{g,1}$ such that φ fixes pointwisely γ, then $\rho_k(\varphi)$ can be computed by 2 L^2-torsion invariants of 2 compact surfaces cutted by γ, i.e.,

$$\rho_k(\varphi) = \rho_k(\varphi_1)\rho_k(\varphi_2).$$

Remark 7.6 In the above case, we can reduce computation to the one for lower genus cases.

$k = 1$ case,

- $\pi(1) = \pi/\Gamma_1 = N_1 \rtimes \mathbb{Z} \cong \mathbb{Z}$
- $GL(2g; \mathbb{Z}\pi(1)) = GL(2g; \mathbb{Z})$.

Here Magnus representation is just

$$r_1 : \mathcal{M}_{g,1} \to Sp(2g : \mathbb{Z}).$$

This Fuglede-Kadison determinant is the usual determinant over $\mathbb{C}[\mathbb{Z}]$. ρ_1 can be computed as follows.

Theorem 7.7 (Lott, Lück, MTK) For any $\varphi \in \mathcal{M}_{g,1}$,

$$\log \rho_1(\varphi) = \int_{S^1} \log|\det(tE - r_1(\varphi))|dt.$$

Remark 7.8 This integration is the Mahler measure for 1-variable polynomials.

By properties of Mahler measure, we can see the following.

Corollary 7.9

$$\log \rho_1(\varphi) = -2 \sum_{i=1}^{2g} \log \max\{1, |\alpha_i|\}$$

Here

$\alpha_1, \ldots, \alpha_{2g}$: the eigenvalues of $r_1(\varphi) \in Sp(2g; \mathbb{Z})$.

In the genus 1 case, 2 eigenvalues of $r_1(\varphi) \in SL(2;\mathbb{Z})$ can be determined by the trace. Now we see the following corollary.

Corollary 7.10 It holds;

$$\log \rho_1(\varphi) = 0 \Leftrightarrow |\mathrm{tr}(r_1(\varphi))| < 2.$$

$-3\pi \log \rho(\varphi)$ is equal to the hyperbolic volume of W_φ, then now we compare the volume with $-3\pi \log \rho_1(\varphi)$.

$\mathrm{tr}(\mathrm{r}_1(\varphi))$	$-3\pi \log \rho_1(\varphi)$	volume
0	0	0
1	0	0
2	0	0
3	18.1412	2.0298
4	24.8240	2.6667
5	29.5334	2.9891
6	33.2270	2.9891
7	36.2825	3.2969
8	38.8948	3.3775

In the genus 1 case, the following holds.

Theorem 7.11 (M-T-K) For any $\varphi \in \mathcal{M}_{1,1}$,

$$\log \rho_2(\varphi) = 0.$$

Theorem 7.12 If $\varphi \in \mathcal{M}_{1,1}$ is a Dehn twist which is a twist along a non-separating curve(=not parallel to the boundary), then

$$\log \rho_k(\varphi) = 0.$$

Problem 7.13 For A such that $|\mathrm{tr}(A)| > 2$, compute ρ_k and investigate its behavior when $k \to \infty$.

Higher genus case :

$k = 2$, Magnus representation

$$r_2 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}H).$$

The mapping class φ_* acts on H trivially,

$$\begin{aligned}\pi(2) =& \ \pi/\Gamma_2 \\ =& N_2 \rtimes \mathbb{Z} \\ =& H \times \mathbb{Z},\end{aligned}$$

then $\mathbb{C}\pi(2)$ is a commutative ring. Hence, we can use the usual determinant.

Under this situation,

Theorem 7.14 (MTK) $\log \rho_2$ can be described by using the Mahler measure for multi-variable polynomials.

By the help of computation by M. Suzuki,

Corollary 7.15 If φ is a BP-map, or a BSCC-map, then

$$\log \rho_2(\varphi) = 0.$$

Remark 7.16 There exists $\varphi \in \mathcal{I}_{g,1}$ such that

$$\log \rho_2(\varphi) \neq 0.$$

Theorem 7.17 If $\varphi \in \mathcal{M}_{g,1}$ is a product of Dehn twists along any disjoint non-separating simple closed curves which are mutually non-homologous, then

$$\log \rho_k(\varphi) = 0.$$

Problem 7.18

$$\lim_{k \longrightarrow \infty} (\rho_k(\varphi)) = \rho(\varphi)?$$