Intersection numbers of twisted cycles and the connection problem for the Fuchsian diferential equations

Workshop on q-Representation
at the center for quantum geometry of moduli space
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Katsuhisa Mimachi
Department of Mathematics,
Tokyo Institute of Technology

Rigid local system

# of irreducible rigid Fuchsian differential systems with 3 singularities on $\mathbb{P}^1$

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	1	3	5	13	20	45	74	142	212	421	588	1004

# of irreducible rigid Fuchsian differential systems

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	2	6	11	28	44	96	157	306	441	857	1117	2032

by Oshima (2008)

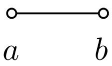
Yokoyama's list (1995)

	rank	# of singularities on $\mathbb{P}^1$	spectrale type
I (HGF)	n	3	$1^n\ ; 1, n-1\ ; 1^n$
I* (Pochhammer)	n	$n-1$	$1, n-1; 1, n-1; \ldots ; 1, n-1$
II	$2n$	3	$1^n, n; 1^n, n; 1, n-1, n$
II*	$2n$	4	$1^n, n; 1^{n-1}, n+1; 1, 2n-1; n, n$
III	$2n+1$	3	$1^{n+1}, n; 1^n, n+1; 1, n, n$
III*	$2n+1$	4	$1^n, n+1; 1^n, n+1; 1, 2n; n, n+1$
IV	6	3	$1^2, 4; 2^3; 1^4, 2$
IV*	6	4	$1^2, 4; 1^2, 4; 2, 4$

Integral representations of the solutions

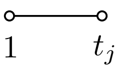
$$\int_C u_C(t)dt_1 \cdots dt_m$$

Diagramatic expression

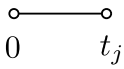


$\Leftrightarrow \quad (a-b)^{\lambda_{ab}}$ or $(b-a)^{\lambda_{ab}}$

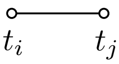
Thus:



$\Leftrightarrow \quad (1-t_j)^{\lambda'_j}$ or $(t_j-1)^{\lambda'_j}$,



$\Leftrightarrow \quad t_j^{\lambda_j}$.



$\Leftrightarrow \quad (t_i-t_j)^{\lambda_{ij}}$ or $(t_j-t_i)^{\lambda_{ij}}$,

(I) Generalized HGF

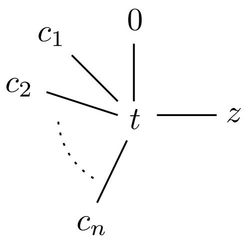
$u(t) = \prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\lambda_{i-1,i}}$

where $t_0 = 1,\ t_{n+1} = z$.

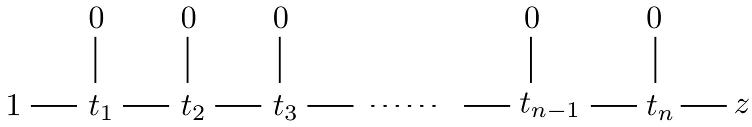
(I*) Pochhammer function

$u(t) = \prod_{j=0}^{n+1} (t - c_j)^{\lambda_j}$

where $c_0 = 0,\ c_{n+1} = z$.

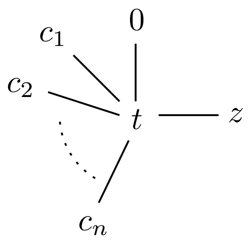


(I) Generalized Hypergeometric function



1^{n+1}	at 0
$1, n$	at 1
1^{n+1}	at ∞

(I*) Pochhammmer function



$1, n-1$	at 0
$1, n-1$	at c_1
$\cdots$	$\cdots$
$1, n-1$	at c_n
$1, n-1$	at ∞

(II)

$1^n, n$	at 0
$1, n-1, n$	at 1
$1^n, n$	at ∞

(II*)

$1^{n-1}, n+1$	at 0
n, n	at 1
$1, 2n-1$	at c
$1^n, n$	at ∞

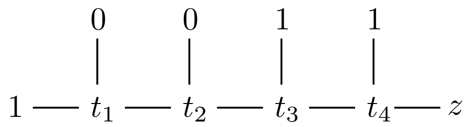
(III)

$1^{n-1}, n$	at 0
$1, n-1, n-1$	at 1
$1^n, n-1$	at ∞

(III*)

$1^n, n+1$	at 0
$n, n+1$	at 1
$1, 2n$	at c
$1^n, n+1$	at ∞

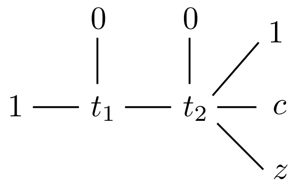
(IV)



(with $\lambda'_1 + \lambda_{12} + \lambda_{23} + \lambda'_3 + 2 = 0$)

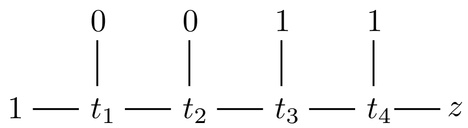
$1^2, 4$	at 0
2^3	at 1
$1^4, 2$	at ∞

(IV*)



$1, 1, 4$	at 0
$1, 1, 4$	at 1
$1, 1, 4$	at c
$2, 4$	at ∞

Remark: Resonance and subsytem.



The resonance $\lambda_{01} + \lambda_{12} + \lambda_{23} + \lambda_{03} + 2 = 0$ induces the subsystem.

$1, 1, 5$	at 0			$1, 1, 4$	at 0
$1, 2, 2, 2$	at 1	$\longrightarrow$	(IV)	$2, 2, 2$	at 1
$1, 1, 1, 2, 2$	at ∞			$1, 1, 1, 1, 2$	at ∞

If the exponent of the irreducible component of the divisor $\widetilde{D} = \pi^{-1}(D)$, where $\pi : \widetilde{(\mathbb{P}^1(\mathbb{C}))^m} \to (\mathbb{P}^1(\mathbb{C}))^m$ is the minimal blow-up along the non-normally crossing loci of D, is an ineger, the irreducible component or the exponent itself is said to be *resonant*.

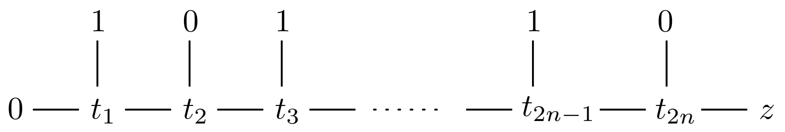
Simpson's list

	rank	spectral type
HGF	n	1^n ; 1^n ; $n-1,1$
Even family	$2n$	1^{2n} ; $n,n-1,1$; n,n
Odd family	$2n+1$	1^{2n+1} ; $n,n,1$; $n+1,n$
Extra case	6	1^6 ; 2^3 ; $4,2$

The even family of rank $2n$ corresponds to the restriction of the Heckman-Opdam HGF of BC_n-type. The Heckman-Opdam HGF of A_n-type corresponds to ${}_{n+1}F_n$.

(Oshima and Shimeno)

Odd family



$1,n,n$ at 0
$n,n+1$ at 1
1^{2n+1} at ∞

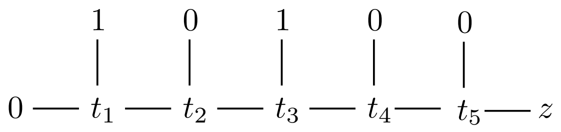
$$\lambda_1'+\lambda_{12}+\lambda_{23}+\lambda_3'+2=0,\quad \lambda_2+\lambda_{23}+\lambda_{34}+\lambda_4+2=0,$$
$$\lambda_3'+\lambda_{34}+\lambda_{45}+\lambda_5'+2=0,\quad \lambda_4+\lambda_{45}+\lambda_{56}+\lambda_6+2=0,$$
$$\cdots \qquad \cdots$$
$$\lambda_{2n-3}'+\ \cdots\ +\lambda_{2n-1}'+2=0,\quad \lambda_{2n-2}+\ \cdots\ +\lambda_{2n}+2=0.$$

Even family

$1,n,n+1$ at 0
$n+1,n+1$ at 1
1^{2n+2} at ∞

$$\lambda_1'+\lambda_{12}+\lambda_{23}+\lambda_3'+2=0,\quad \lambda_2+\lambda_{23}+\lambda_{34}+\lambda_4+2=0,$$
$$\cdots \qquad \cdots$$
$$\lambda_{2n-3}'+\ \cdots\ +\lambda_{2n-1}'+2=0,\quad \lambda_{2n-2}+\ \cdots\ +\lambda_{2n}+2=0$$
$$\lambda_{2n-1}'+\ \cdots\ +\lambda_{2n+1}'+2=0.$$

Exitra



$1,2,3$ at 0
$3,3$ at 1
1^6 at ∞

$$\lambda_1+\lambda_{12}+\lambda_{45}+\lambda_5+4=\lambda_2+\lambda_{23}+\lambda_{45}+\lambda_4+2=0$$

Remark. The rank of the twisted homology group $H_n(T, \mathcal{L})$ in case of

$$\begin{array}{ccccccccccccccc} & & 1 & & 0 & & 1 & & & & 1 & & 0 & & \\ & & | & & | & & | & & & & | & & | & & \\ 0 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & — & z \end{array} \qquad \text{or}$$

n : even

$$\begin{array}{ccccccccccccccc} & & 1 & & 0 & & 1 & & & & 0 & & 1 & & \\ & & | & & | & & | & & & & | & & | & & \\ 0 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & — & z \end{array}$$

n : odd

is a_{n+2}. Here a_n is the Fibonacci number: $a_1 = a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, a_9 = 34, a_{10} = 55, a_{11} = 89, \ldots$.

Connection formulas

(1) **Generalized hypergeometric function ${}_{n+1}F_n$.**

$$f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s\neq i} \frac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s\neq j} \frac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z)\,,$$

where $f_i^{(0)}(z) = (-z)^{1-\beta_i}(1 + O(z)), f_i^{(\infty)}(z) = (-z)^{-\alpha_i}(1 + O(z^{-1}))$.

$$f_1^{(1)}(z) = \sum_{j=1}^{n+1} \prod_{s\neq i} \frac{\Gamma(1 + \sum_{s=1}^{n} \beta_s - \sum_{s=1}^{n+1} \alpha_s) \prod_{\substack{1\le s\le n+1\\ s\neq i}} \Gamma(\beta_j - \beta_s)}{\prod_{1\le s\le n+1} \Gamma(\beta_j - \alpha_s)} \times f_j^{(0)}(z)\,,$$

where $f_i^{(0)}(z) = (-z)^{1-\beta_i}(1 + O(z)), f_1^{(1)}(z) = (1 - z)^{\sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n+1} \alpha_i}(1 + O(1 - z))$.

(2) **Even family of rank=4** (joint work with Haraoka):

$$\int_D t_1^{\lambda_1}(t_1 - 1)^{\lambda_2}(t_1 - t_2)^{\lambda_3} t_2^{\lambda_4}(t_2 - t_3)^{\lambda_5}(t_3 - 1)^{\lambda_6}(t_3 - z)^{\lambda_7} dt_1 dt_2 dt_3$$

$(\lambda_{2356} + 2 = 0,\ \lambda_{ij\cdots k} = \lambda_i + \lambda_j + \cdots + \lambda_k)$

$$F_1^{(0)}(z) = (-z)^{\lambda_{13457}+3}(1 + O(z)),$$

$$\begin{aligned} F_1^{(\infty)}(z) &= (-z)^{\lambda_{1234567}+3}\left(1 + O(z^{-1})\right), \\ F_2^{(\infty)}(z) &= (-z)^{\lambda_{34567}+2}\left(1 + O(z^{-1})\right), \\ F_3^{(\infty)}(z) &= (-z)^{\lambda_{567}+1}\left(1 + O(z^{-1})\right), \\ F_4^{(\infty)}(z) &= \left(1 + O(z^{-1})\right). \end{aligned}$$

$$F_1^{(0)}(z) = \sum_{j=1}^{4} p_{1j} F_j^{(\infty)}(z),$$

where

$$\begin{aligned} p_{11} &= \frac{\Gamma(1 + \lambda_{12}, 1 + \lambda_{14}, 2 + \lambda_{13}, 1 + \lambda_{1234}, 4 + \lambda_{13457})}{\Gamma(1 + \lambda_1, 2 + \lambda_{123}, 2 + \lambda_{134}, 2 + \lambda_{147}, 3 + \lambda_{12345})}, \\ p_{12} &= \frac{\Gamma(1 + \lambda_{34}, 2 + \lambda_{13}, 2 + \lambda_{3456}, 4 + \lambda_{1357}, -1 - \lambda_{12})}{\Gamma(1 + \lambda_3, 2 + \lambda_{134}, 2 + \lambda_{345}, 3 + \lambda_{34567}, -\lambda_2)}, \\ p_{13} &= \frac{\Gamma(1 + \lambda_{56}, 2 + \lambda_{13}, 4 + \lambda_{13457}, -1 - \lambda_{34}, -2 - \lambda_{1234})}{\Gamma(1 + \lambda_1, 1 + \lambda_5, 2 + \lambda_{567}, -\lambda_2, -\lambda_4)}, \\ p_{14} &= \frac{\Gamma(2 + \lambda_{13}, 4 + \lambda_{13457}, -2 - \lambda_{3456}, -1 - \lambda_{56}, -1 - \lambda_{14})}{\Gamma(1 + \lambda_3, 1 + \lambda_7, 2 + \lambda_{123}, -\lambda_4, -\lambda_6)}, \end{aligned}$$

with $\Gamma(a_1, a_2, \cdots, a_m) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_m)$.

$_{n+1}F_n$ case

Derivation by use of the intersection number.

$H_n^{\mathrm{lf}}(T,\mathcal{L})$ or $H_n(T,\mathcal{L})$, where $\mathcal{L}$ is determined by

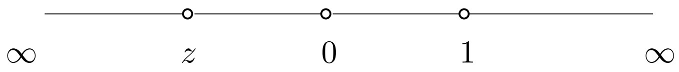
$$u(t)=\prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i}\prod_{i=1}^{n+1}(t_i-t_{i-1})^{\beta_i-\alpha_i-1},\quad (\beta_{n+1}=1, t_0=1, t_{n+1}=z),$$

on

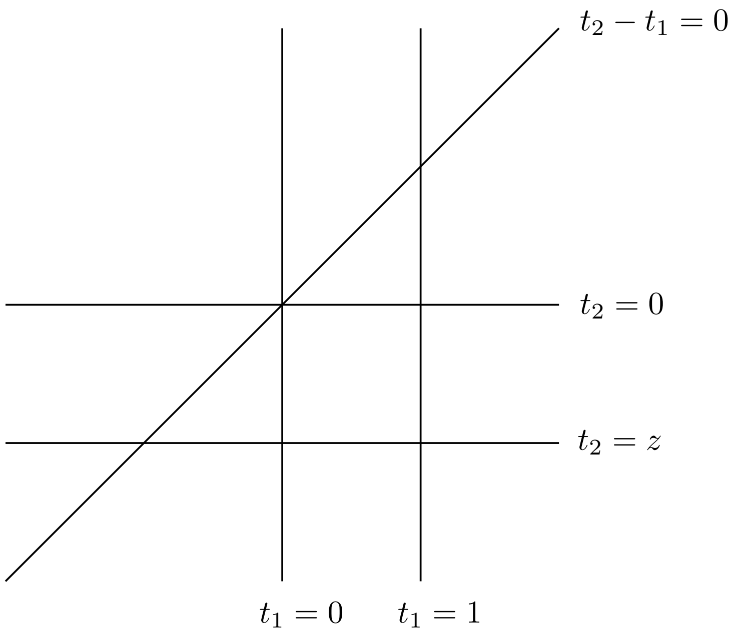
$$T=\mathbb{C}^n\setminus\cup_{i=1}^{n}\{\,t_i=0\,\}\cup\cup_{i=1}^{n+1}\{\,t_i-t_{i-1}=0\,\}.$$

In what follows, z is fixed to be $\infty<z<0$.

$n=1$



$n=2$

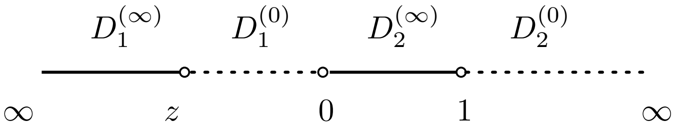


Bases of $H_n^{\mathrm{lf}}(T, \mathcal{L})$:

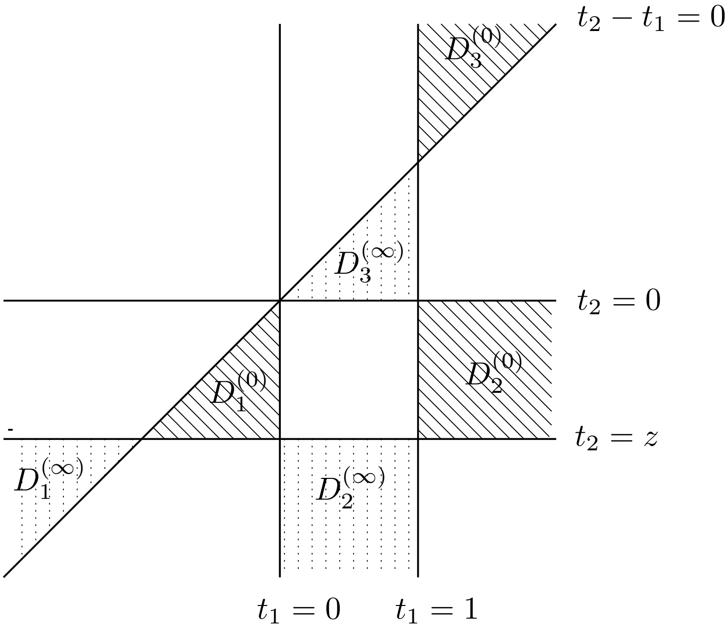
$$\left\{ D_1^{(0)}, D_2^{(0)}, \dots, D_{n+1}^{(0)} \;\middle|\; D_i^{(0)} = \begin{pmatrix} \infty < z < t_n < \cdots < t_i < 0 \\ 1 < t_1 < \cdots < t_{i-1} < \infty \end{pmatrix} \right\},$$

$$\left\{ D_1^{(\infty)}, D_2^{(\infty)}, \dots, D_{n+1}^{(\infty)} \;\middle|\; D_i^{(\infty)} = \begin{pmatrix} \infty < t_i < \cdots < t_n < z \\ 0 < t_{i-1} < \cdots < t_1 < 1 \end{pmatrix} \right\}.$$

$n = 1$



$n = 2$



$\Longrightarrow \exists\, c_{ij}$ such that

$$D_i^{(\infty)} = \sum_{1 \le j \le n+1} c_{ij}\, D_j^{(0)}$$

On the other hand,

$$\int_{D_i^{(0)}} u_{D_i^{(0)}}(t)\,dt_1\cdots dt_n = \prod_{\substack{1\le s\le n+1\\ s\ne i}} B(\alpha_s-\beta_i+1,\beta_s-\alpha_s)\times f_i^{(0)}(z),$$

$$\int_{D_i^{(\infty)}} u_{D_i^{(\infty)}}(t)\,dt_1\cdots dt_n = \prod_{\substack{1\le s\le n+1\\ s\ne i}} B(\alpha_i-\beta_s+1,\beta_s-\alpha_s)\times f_i^{(\infty)}(z).$$

For $u(t)=\prod_i f_i(t)^{\alpha_i}$, $u_D(t)=\prod_i(\epsilon_i\, f_i(t))^{\alpha_i}$, where $\epsilon_i=\pm$ is determined so that $\epsilon_i\, f_i(t)>0$ on D.

Intersection form (Intersection numbers)

The map

$$\mathrm{reg}\ :\ H_m^{\mathrm{lf}}(T,\mathcal{L})\ \longrightarrow\ H_m(T,\mathcal{L})$$

is defined as an inverse of the natural map

$$\iota\ :\ H_m(T,\mathcal{L})\ \longrightarrow\ H_m^{\mathrm{lf}}(T,\mathcal{L}).$$

To define the intersection numbers for $C, C'\in H_m^{\mathrm{lf}}(T,\mathcal{L})$, we first regularize one of them, secondly compute the intersection number of the consequent cycles and finally sum up them. Actually, the *intersection form*

$$\langle\quad,\quad\rangle : H_n^{\mathrm{lf}}(T,\mathcal{L})\times H_n^{\mathrm{lf}}(T,\mathcal{L})\longrightarrow\mathbb{C}$$

is the Hermitian form defined by

$$(C,C')\longmapsto\langle C,C'\rangle=\sum_{\rho,\,\sigma}a_\rho\,\overline{a'_\sigma}\sum_{t\in\rho\cap\sigma}I_t(\rho,\sigma)v_\rho(t)\overline{v'_\sigma(t)}/|u|^2,$$

for C, $C'\in H_m^{\mathrm{lf}}(T,\mathcal{L})$, if

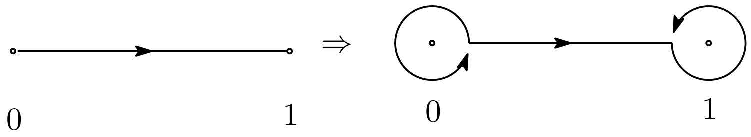
$$\mathrm{reg}\ C=\sum_\rho a_\rho\,\rho\otimes v_\rho,\qquad C'=\sum_\sigma a'_\sigma\,\sigma\otimes v'_\sigma,$$

where $a_\rho, a'_\sigma\in\mathbb{C}$, ρ,σ:n-simplex, v_ρ, v'_σ: a section of $\mathcal{L}$ on ρ, σ, $^-$: the complex conjugation, $I_t(\rho,\sigma)$: the topological intersection number of ρ and σ at t.

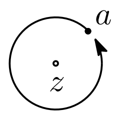
The value $\langle C,C'\rangle$ is called the *intersection number* of C and C' and written also by $C\bullet C'$

Example of regularization. $T = \mathbb{C}\backslash\{0,1\}$, $u(t) = t^\alpha(1-t)^\beta$.

$$\overrightarrow{(0,1)} \quad \Rightarrow \quad \mathrm{reg}\,\overrightarrow{(0,1)} = \left\{ \frac{1}{d_\alpha} S(\epsilon\,;0) + \overrightarrow{[\epsilon, 1-\epsilon]} - \frac{1}{d_\beta} S(1-\epsilon\,;1) \right\}$$



Here $d_a = e(a) - 1, \quad e(a) = \exp(2\pi\sqrt{-1}a)$. The symbol $S(a\,;z)$ stands for the positively oriented circle centered at the point z with starting and ending point a, ϵ is a small positive number and the argument of each factor of $u(t)$ on the oriented circle $S(\epsilon\,;0)$ or $S(1-\epsilon\,;1)$ is defined so that $\arg t$ takes value from 0 to 2π on $S(\epsilon\,;0)$, and $\arg(1-t)$ from 0 to 2π on $S(1-\epsilon\,;1)$.

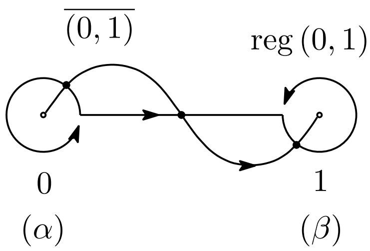


Examples of intersection numbers.

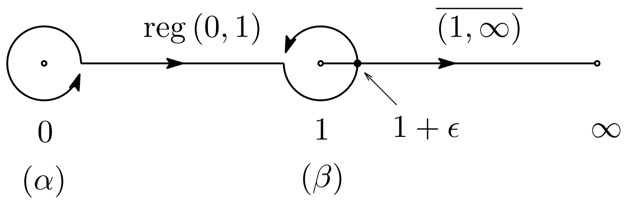
$$\overrightarrow{(0,1)} \bullet \overrightarrow{(0,1)} = -\frac{1}{d_\alpha} - 1 + \frac{-1}{d_\beta}$$

$$= -\frac{d_{\alpha+\beta}}{d_\alpha d_\beta} = -\frac{s(\alpha+\beta)}{s(\alpha)s(\beta)},$$

where $s(a) = \sin(\pi a)$



$$\overrightarrow{(0,1)} \bullet \overrightarrow{(1,\infty)} = \frac{e(\beta/2)}{e(\beta)-1}$$



Connection coefficients in terms of intersection numbers

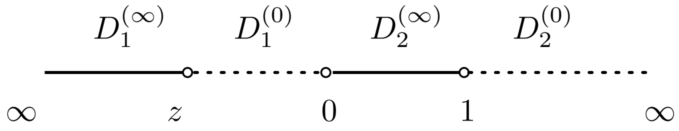
$$D_i^{(\infty)} = \sum_{1\le j\le n+1} c_{ij}\, D_j^{(0)}, \qquad C = (c_{ij}),$$

$$\begin{pmatrix} D_1^{(\infty)} \\ \vdots \\ \vdots \\ D_{n+1}^{(\infty)} \end{pmatrix} \bullet (D_1^{(0)}, \dots, D_{n+1}^{(0)}) = C \begin{pmatrix} D_1^{(0)} \\ \vdots \\ \vdots \\ D_{n+1}^{(0)} \end{pmatrix} \bullet (D_1^{(0)}, \dots, D_{n+1}^{(0)}),$$

$$\begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} = C \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix},$$

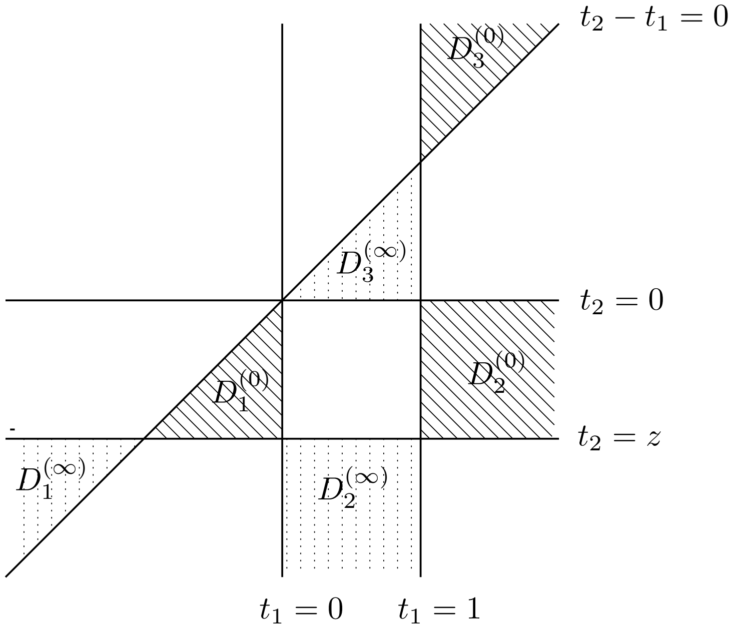
$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}.$$

$n = 1$



$\Longrightarrow$ $D_1^{(0)} \bullet D_2^{(0)} = 0$
$D_2^{(0)} \bullet D_1^{(0)} = 0$

$n = 2$



$\Longrightarrow$ $D_i^{(0)} \bullet D_j^{(0)} = 0$
$(i \neq j)$

$$C=\begin{pmatrix} D_1^{(\infty)}\bullet D_1^{(0)} & \cdots & D_1^{(\infty)}\bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)}\bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)}\bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)}\bullet D_1^{(0)} & \cdots & D_1^{(0)}\bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)}\bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)}\bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$D_i^{(0)}\bullet D_j^{(0)}=\delta_{ij}\left(\frac{\sqrt{-1}}{2}\right)^n\prod_{\substack{1\le s\le n+1\\ s\neq j}}\frac{\sin(\beta_s-\beta_j)}{\sin(\beta_s-\alpha_s)\sin(\alpha_s-\beta_j)}$,

$D_i^{(\infty)}\bullet D_j^{(0)}=\left(\frac{\sqrt{-1}}{2}\right)^n\frac{1}{\sin(\beta_j-\alpha_i)}\prod_{\substack{1\le s\le n+1\\ s\neq i,j}}\frac{1}{\sin(\beta_s-\alpha_s)}$.

$$\Longrightarrow \quad c_{ij}=\frac{D_i^{(\infty)}\bullet D_j^{(0)}}{D_j^{(0)}\bullet D_j^{(0)}}=\frac{\sin(\beta_i-\alpha_i)}{\sin(\beta_j-\alpha_i)}\prod_{\substack{1\le s\le n+1\\ s\neq j}}\frac{\sin(\alpha_s-\beta_j)}{\sin(\beta_s-\beta_j)}$$

$$\Longrightarrow \quad f_i^{(\infty)}(z)=\sum_{j=1}^{n+1}\prod_{s\neq i}\frac{\Gamma(\alpha_i-\alpha_s+1)}{\Gamma(\beta_j-\alpha_s)}\prod_{s\neq j}\frac{\Gamma(\beta_j-\beta_s)}{\Gamma(\alpha_i-\beta_s+1)}\times f_j^{(0)}(z)$$

We recall that

$$\int_{D_i^{(0)}} u_{D_i^{(0)}}(t)\,dt_1\cdots dt_n=\prod_{\substack{1\le s\le n+1\\ s\neq i}} B(\alpha_s-\beta_i+1,\beta_s-\alpha_s)\times f_i^{(0)}(z),$$
$$\int_{D_i^{(\infty)}} u_{D_i^{(\infty)}}(t)\,dt_1\cdots dt_n=\prod_{\substack{1\le s\le n+1\\ s\neq i}} B(\alpha_i-\beta_s+1,\beta_s-\alpha_s)\times f_i^{(\infty)}(z).$$