Kernel functions for van Diejen's q-difference operators

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0 General remarks on kernel functions

0.1 Kernel function $\Phi(x;y)$ for a pair of operators $(\mathcal{A}_x, \mathcal{B}_y)$

Let $\Phi(x;y)$ be a meromorphic function in $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, and consider two operators $\mathcal{A}_x$ and $\mathcal{B}_y$ which act on meromorphic functions in x and in y, respectively. We say that $\Phi(x;y)$ is a *kernel function* for the pair $(\mathcal{A}_x, \mathcal{B}_y)$ if it satisfies a functional equation of the form

$$\mathcal{A}_x \Phi(x;y) = \mathcal{B}_y \Phi(x;y).$$

In the theory of Macdonald polynomials, certain explicit kernel functions play crucial roles in eigenfunction expansions and integral representations.

Eigenfunction expansion

$$\Phi(x;y) = \sum_k f_k(x)\, g_k(y), \quad \mathcal{B}_y\, g_k(y) = g_k(y)\lambda_k \quad \Longrightarrow \quad \mathcal{A}_x f_k(x) = f_k(x)\, \lambda_k.$$

Integral representation

$$\varphi(x) = \int \Phi(x;y)\psi(y) d\mu(y), \quad \mathcal{B}_y^*\, \psi(y) = \psi(y)\lambda \quad \Longrightarrow \quad \mathcal{A}_x \varphi(x) = \varphi(x)\, \lambda.$$

0.2 Macdonald polynomials of type A

In order to clarify the idea, we first look at the role of kernel functions in the theory of Macdonald polynomials of type A.

Consider the following q-difference operator $\mathcal{D}_x = \mathcal{D}_x^{(q,t)}$ in the variables $x = (x_1, \ldots, x_m)$:

$$\mathcal{D}_x = \mathcal{D}_x^{(q,t)} = \sum_{i=1}^{m} \prod_{1 \le j \le m;\, j \ne i} \frac{tx_i - x_j}{x_i - x_j}\, T_{q,x_i};$$
$$T_{q,x_i} f(x_1, \ldots, x_m) = f(x_1, \ldots, qx_i, \ldots, x_m).$$

The *Macdonald polynomials* $P_\lambda(x) = P_\lambda(x|q,t)$, parameterized by the partitions λ with $l(\lambda) \le m$, are characterized as symmetric polynomials in x such that

(1) $P_\lambda(x) = m_\lambda(x) +$ lower terms w.r.t. the dominance ordering,

(2) $\mathcal{D}_x P_\lambda(x) = P_\lambda(x)\, d_\lambda, \qquad d_\lambda = \sum_{i=1}^{m} t^{m-i} q^{\lambda_i},$

where $m_\lambda(x)=\sum_{\mu\in\mathfrak{S}_m\lambda}x^\mu$ stands for the monomial symmetric function of monomial type λ.

Furthermore, $\mathcal{D}_x$ admits a commuting family $\mathcal{D}_x^{(r)}$ $(r=0,1,\ldots,m)$ of higher order q-difference operators such that $\mathcal{D}_x^{(1)}=\mathcal{D}_x$:

$$\mathcal{D}_x^{(r)}=t^{\binom{r}{2}}\sum_{|I|=r}\prod_{i\in I;j\notin I}\frac{tx_i-x_j}{x_i-x_j}\prod_{i\in I}T_{q,x_i}\qquad(r=0,1,\ldots,m).$$

These *Macdonald-Ruijsenaars q-difference operators* are simultaneously diagonalized by the the Macdonald polynomials. In terms of the generating function $\mathcal{D}_x(u)=\sum_{r=0}^{m}(-u)^r\mathcal{D}_x^{(r)}$, one has

$$\mathcal{D}_x(u)\,P_\lambda(x)=P_\lambda(x)d_\lambda(u),\qquad d_\lambda(u)=\prod_{i=1}^m(1-ut^{m-i}q^{\lambda_i}).$$

0.3 Kernel function of Cauchy type

Assuming that $|q|<1$, we define a meromorphic function $\Phi(x;y|q,t)$ in the two sets of variables $x=(x_1,\ldots,x_m)$, $y=(y_1,\ldots,y_n)$ by

$$\Phi(x;y|q,t)=\prod_{j=1}^{m}\prod_{l=1}^{n}\frac{(tx_jy_l;q)_\infty}{(x_jy_l;q)_\infty}\qquad\text{(kernel function of \textit{Cauchy type})},$$

where $(a;q)_\infty=\prod_{k=0}^\infty(1-aq^k)$. This function $\Phi(x;y|q,t)$ is expanded as follows in terms of Macdonald polynomials in x variables and y variables:

$$\Phi(x;y|q,t)=\sum_{l(\lambda)\le\min\{m,n\}}b_\lambda(q,t)\;P_\lambda(x|q,t)\;P_\lambda(y|q,t).$$

This identity corresponds to the functional equation

$$\mathcal{D}_x^{(q,t)}\Phi(x;y|q,t)-t^{m-n}\mathcal{D}_y^{(q,t)}\Phi(x;y|q,t)=\frac{1-t^{m-n}}{1-t}\Phi(x;y|q,t)$$

for the q-difference operator $\mathcal{D}_x=\mathcal{D}_x^{(q,t)}$. Apart from the question of evaluating the coefficients $b_\lambda(q,t)$, this functional equation already guarantees the existence of an expansion formula as described above. Note also that the kernel function $\Phi(x;y|q,t)$ furthermore intertwines the whole commuting family of Macdonald-Ruijsenaars operators:

$$\mathcal{D}_x(u)\Phi(x;y|q,t)=(u;t)_{m-n}\,\mathcal{D}_y(t^{m-n}u)\Phi(x;y|q,t).$$

0.4 Kernel function of dual Cauchy type

The function

$$\Psi(x;y) = \prod_{j=1}^{m}\prod_{l=1}^{n} (x_j - y_l) \qquad \text{(kernel function of } \textit{dual Cauchy type}\text{)}$$

is expanded in terms of Macdonald polynomials as follows:

$$\Psi(x;y) = \sum_{\lambda \subset (n^m)} (-1)^{|\lambda^*|} \, P_\lambda(x|q,t) \, P_{\lambda^*}(y|t,q),$$

where $\lambda^* = (m - \lambda'_m, m - \lambda'_{m-1}, \ldots, m - \lambda'_1)$ is the partition representing the complement of λ in the $m \times n$ rectangle (n^m). This expansion formula corresponds to the functional equation

$$(1-t)\mathcal{D}_x^{(q,t)}\Psi(x;y) - (1-q)\mathcal{D}_y^{(t,q)}\Psi(x;y) = (1-t^m q^n)\Psi(x;y).$$

1 Koornwinder polynomials

Koornwinder polynomials (Koornwinder, 1992)
$\cdots$ multivariable generalization of Askey-Wilson polynomials to type BC_m.

1.1 Koornwinder's q-difference operator $\mathcal{D}_x$

We consider the q-difference operator $\mathcal{D}_x = \mathcal{D}_x^{(a,b,c,d|q,t)}$ in m variables $x = (x_1, \ldots, x_m)$, depending on six parameters $(a, b, c, d|q, t)$:

$$\begin{aligned}\mathcal{D}_x &= \sum_{i=1}^{m} A_{i,+}(x)(T_{q,x_i} - 1) + \sum_{i=1}^{m} A_{i,-}(x)(T_{q,x_i}^{-1} - 1) \\ &= \sum_{i=1}^{m} \left(A_{i,+}(x)T_{q,x_i} + A_{i,-}(x)T_{q,x_i}^{-1}\right) - \sum_{i=1}^{m} \left(A_{i,+}(x) + A_{i,-}(x)\right),\end{aligned}$$

where $A_{i,\pm}(x) = A_{i,\pm}(x; a, b, c, d|q, t)$ are defined by

$$\begin{aligned}A_{i,+}(x) &= \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(abcdq^{-1})^{\frac{1}{2}}(1-x_i^2)(1-qx_i^2)} \prod_{1\le j\le m;\ j\ne i} \frac{(1-tx_i/x_j)(1-tx_ix_j)}{t(1-x_i/x_j)(1-x_ix_j)}, \\ A_{i,-}(x) &= A_{i,+}(x^{-1}) \qquad (i = 1, \ldots, m).\end{aligned}$$

In the multiplicative notation for the sine function

$$\langle z\rangle = z^{\frac{1}{2}} - z^{-\frac{1}{2}} = 2\sqrt{-1}\sin(\pi\zeta), \qquad z = e(\zeta) = e^{2\pi\sqrt{-1}\zeta},$$

the coefficients $A_{i,+}(x)$ are expressed as

$$A_{i,+}(x) = \frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle} \prod_{1\le j\le m; j\ne i} \frac{\langle tx_i/x_j\rangle\langle tx_ix_j\rangle}{\langle x_i/x_j\rangle\langle x_ix_j\rangle} \quad (i = 1, \ldots, m).$$

1.2 Koornwinder polynomials

We denote by $P = P(C_m)$ the weight lattice of type C_m, and by P^+ the corresponding cone of dominant integral weights:

$$P = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_m, \quad P^+ = \{\lambda = \textstyle\sum_{i=1}^m \lambda_i\varepsilon_i \in P \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0\}.$$

The cone P^+ is identified with the set of all partitions λ with $l(\lambda) \le m$. Also, we denote by $W = W(C_m) = \{\pm 1\}^m \rtimes \mathfrak{S}_m$ the Weyl group (hyperoctahedral group).

For generic parameters $(a,b,c,d|q,t)$, the *Koornwinder polynomials* $P_\lambda(x) = P_\lambda(x;a,b,c,d|q,t)$ $(\lambda \in P^+)$ are characterized as W-invariant Laurent polynomials in x such that

(1) $P_\lambda(x) = m_\lambda(x) +$ lower terms w.r.t. the dominance ordering,

(2) $\mathcal{D}_x P_\lambda(x) = P_\lambda(x) d_\lambda, \quad d_\lambda = \sum_{i=1}^m \left(\alpha t^{m-i}(q^{\lambda_i}-1)+\alpha^{-1}t^{-m+i}(q^{-\lambda_i}-1)\right),$

where $m_\lambda(x) = \sum_{\mu\in W.\lambda} x^\mu$ stands for the orbit sum of the monomial x^λ, and $\alpha = (abcdq^{-1})^{\frac{1}{2}}$.

We take the coefficient field $\mathbb{K} = \mathbb{Q}(a^{\frac{1}{2}}, b^{\frac{1}{2}}, c^{\frac{1}{2}}, d^{\frac{1}{2}}, q^{\frac{1}{2}}, t^{\frac{1}{2}})$, regarding the square roots of the six parameters as indeterminates. The Koornwinder polynomials then form a $\mathbb{K}$-basis of the ring $\mathbb{K}[x^{\pm 1}]^W = \mathbb{K}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]^W$ of W-invariant Laurent polynomials:

$$\mathbb{K}[x^{\pm}]^W = \bigoplus_{\lambda\in P^+} \mathbb{K}\, P_\lambda(x).$$

1.3 Orthogonality

Assuming that $|q|<1$, we define the weight function $w(x) = w(x;a,b,c,d|q,t)$ by setting $w(x) = w_+(x)w_+(x^{-1})$ where

$$w_+(x) = \prod_{i=1}^m \frac{(x^2;q)_\infty}{(ax_i,bx_i,cx_i,dx_i;q)_\infty} \prod_{1\le i<j\le m} \frac{(x_i/x_j, x_ix_j;q)_\infty}{(tx_i/x_j, tx_ix_j;q)_\infty}.$$

When $\max\{|a|,|b|,|c|,|d|,|t|\} < 1$, the Koornwinder polynomials satisfy the orthogonality relation

$$\int_{\mathbb{T}^m} P_\lambda(x)P_\mu(x)w(x)\frac{dx_1\cdots dx_m}{x_1\cdots x_m} = 0 \qquad (\lambda,\mu\in P^+; \lambda\neq\mu)$$

where $\mathbb{T}^m = \left\{x\in(\mathbb{C}^*)^m \;\middle|\; |x_i| = 1 \;\; (i=1,\ldots,m)\right\}$.

This orthogonality follows from the fact that $\mathcal{D}_x$ is formally selfadjoint with respect to the weight function $w(x)$. Note also that the leading coefficient $A_{1,+}(x)$ of $\mathcal{D}_x$ is recovered from the positive part $w_+(x)$ of $w(x)$ by

$$A_{1,+}(x) = \frac{\langle ax_1\rangle\langle bx_1\rangle\langle cx_1\rangle\langle dx_1\rangle}{\langle x_1^2\rangle\langle qx_1^2\rangle} \prod_{2\le j\le m} \frac{\langle tx_1/x_j\rangle\langle tx_1x_j\rangle}{\langle x_1/x_j\rangle\langle x_1x_j\rangle} = \text{const.}\frac{T_{q,x_1}w_+(x)}{w_+(x)}.$$

2 q-Representation theoretic aspects

2.1 Affine Hecke algebra

The Koornwinder polynomials can be formulated in terms of (double) affine Hecke algebras. Consider the Hecke algebras

$$\mathcal{H}(W^{\mathrm{aff}}) = \mathbb{K}\langle T_0, T_1, \ldots, T_m\rangle \supset \mathcal{H}(W) = \mathbb{K}\langle T_1, \ldots, T_m\rangle$$

of type $C_m^{(1)}$ and of type C_m, imposing the quadratic relations

$$(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}}) = 0 \qquad (i = 0, 1, \ldots, m)$$

with three unequal parameters $t_0, t_1 = \cdots = t_{m-1} = t, t_m$. Note that the braid relations of degree four

$$T_0T_1T_0T_1 = T_1T_0T_1T_0, \quad T_{m-1}T_mT_{m-1}T_m = T_mT_{m-1}T_mT_{m-1}$$

are imposed on the pairs of indices $(0, 1)$ and $(m-1, m)$ when $m \geq 2$.

The affine Hecke algebra $\mathcal{H}(W^{\mathrm{aff}})$ acts on the $\mathbb{K}$-algebra of Laurent polynomials $\mathcal{A} = \mathbb{K}[x^{\pm}]$ through the Lusztig operators. In fact there is a two-parameter family of $\mathbb{K}$-algebra homomorphisms

$$\rho_{u_0,u_m} : \mathcal{H}(W^{\mathrm{aff}}) \to \mathcal{D}_{q,x}[W] = \mathbb{K}(x)[\tau^{\pm 1}; W] \qquad (\tau = (\tau_1, \ldots, \tau_m); \tau_i = T_{q,x_i})$$

from the affine Hecke algebra to the ring of q-difference-reflection operators such that

$$\begin{aligned}
\rho_{u_0,u_m}(T_0) &= t_0^{\frac{1}{2}} + \frac{\langle t_0 q/x_1^2\rangle + \langle u_0\rangle}{\langle q/x_1^2\rangle}(s_0 - 1),\\
\rho_{u_0,u_m}(T_i) &= t^{\frac{1}{2}} + \frac{\langle t x_i/x_{i+1}\rangle}{\langle x_i/x_{i+1}\rangle}(s_i - 1) \qquad (i = 1, \ldots, m-1),\\
\rho_{u_0,u_m}(T_m) &= t_m^{\frac{1}{2}} + \frac{\langle t_m x_m^2\rangle + \langle u_m\rangle}{\langle x_m^2\rangle}(s_m - 1).
\end{aligned}$$

The four parameters (t_0, t_m, u_0, u_m) are related to the Askey-Wilson parameters (a, b, c, d) through

$$a = t_m^{\frac{1}{2}} u_m^{\frac{1}{2}}, \quad b = -t_m^{\frac{1}{2}} u_m^{\frac{1}{2}}, \quad c = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, \quad d = -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}.$$

2.2 q-Dunkl operators

In this affine Hecke algebra $\mathcal{H}(W^{\mathrm{aff}})$, we define the *q-Dunkl operators* or the *Cherednik operators* $Y_1, \ldots, Y_m$ by

$$Y_i = T_i T_{i+1} \cdots T_m T_{m-1} \cdots T_0 T_1^{-1} \cdots T_{i-1}^{-1} \qquad (i = 1, \ldots, m).$$

These elements commute with each other, and the monomials $Y^\mu = Y_1^{\mu_1} \cdots Y_m^{\mu_m}$ ($\mu \in P$) form a commutative $\mathbb{K}$-subalgebra

$$\mathbb{K}[Y^{\pm 1}] = \bigoplus_{\mu \in P} \mathbb{K} Y^\mu \subset \mathcal{H}(W^{\mathrm{aff}})$$

of the affine Hecke algebra. Let $\{T_w\}_{w \in W}$ be the $\mathbb{K}$-basis of the finite dimensional Hecke algebra $\mathcal{H}(W)$ defined by setting $T_w = T_{i_1} \cdots T_{i_l}$ for any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$. Then we obtain a $\mathbb{K}$-basis $\{Y^\mu T_w \mid \mu \in P, w \in W\}$ for the affine Hecke algebra:

$$\mathcal{H}(W^{\mathrm{aff}}) = \bigoplus_{\mu \in P; w \in W} \mathbb{K}\, Y^\mu T_w = \bigoplus_{w \in W} \mathbb{K}[Y^{\pm 1}] T_w = \bigoplus_{\mu \in P} Y^\mu\, \mathcal{H}(W).$$

Furthermore the center of the affine Hecke algebra coincides with the ring of W-invariant Laurent polynomials of the q-Dunkl operators (Bernstein's theorem):

$$\mathcal{Z}\mathcal{H}(W^{\mathrm{aff}}) = \mathbb{K}[Y^{\pm 1}]^W.$$

On the ring of Laurent polynomials $\mathcal{A} = \mathbb{K}[x^{\pm 1}]$, the elements of the commutative subalgebra $\mathbb{K}[Y^{\pm 1}]$ of the affine Hecke algebra $\mathcal{H}(W^{\mathrm{aff}})$ are simultaneously diagonalized by the *nonsymmetric Koornwinder polynomials* $E_\mu(x)$ ($\mu \in P$). Furthermore $\mathcal{A} = \mathbb{K}[x^{\pm 1}]$ decomposes into the direct sum of irreducible $\mathcal{H}(W^{\mathrm{aff}})$-submodules $V(\lambda)$ parametrized by $\lambda \in P^+$:

$$\mathcal{A} = \mathbb{K}[x^{\pm}] = \bigoplus_{\lambda \in P^+} V(\lambda), \qquad V(\lambda) = \bigoplus_{\mu \in W.\lambda} \mathbb{K}\, E_\mu(x).$$

Each $V(\lambda)$ has a one-dimensional $\mathbb{K}$-subspace of $\mathcal{H}(W)$-invariants (W-invariants) whose basis is given by the Koornwinder polynomial $P_\lambda(x)$:

$$\mathcal{A}^W = \mathcal{A}^{\mathcal{H}(W)} = \bigoplus_{\lambda \in P^+} V(\lambda)^{\mathcal{H}(W)}; \quad V(\lambda)^{\mathcal{H}(W)} = \mathbb{K}\, P_\lambda(x).$$

In this sense the Koornwinder polynomials are thought of as zonal spherical functions for the pair $(\mathcal{H}(W^{\mathrm{aff}}), \mathcal{H}(W))$ of the affine Hecke algebra relative to the finite dimensional Hecke algebra.

2.3 Commuting family of q-difference operators

By restricting the action of the central elements $f(Y)$, $f \in \mathbb{K}[\xi^{\pm 1}]^W$, to the subring of W-invariant Laurent polynomials $\mathcal{A}^W = \mathbb{K}[x^{\pm 1}]^W$, we obtain a commuting family of W-invariant q-difference operators with rational coefficients:

$$\mathcal{D}_f = f(Y)\big|_{\mathbb{K}[x^{\pm 1}]^W} : \ \mathbb{K}[x^{\pm 1}]^W \to \mathbb{K}[x^{\pm 1}]^W \qquad (f \in \mathbb{K}[\xi^{\pm 1}]^W).$$

If we take

$$f(\xi) = \sum_{i=1}^{m} (\xi_i + \xi_i^{-1} - \alpha t^{m-i} - \alpha^{-1} t^{-m+i}) \in \mathbb{K}[\xi^{\pm 1}]^W, \quad \alpha = (t_0 t_m)^{\frac{1}{2}} = (abcdq^{-1})^{\frac{1}{2}}$$

for f in particular, it turns out that the corresponding q-difference operator $\mathcal{D}_f$ is precisely the q-difference operator of Koornwinder

$$\mathcal{D}_f = \mathcal{D}_x = \sum_{i=1}^{m} A_{i,+}(x)(T_{q,x_i} - 1) + \sum_{i=1}^{m} A_{i,-}(x)(T_{q,x_i}^{-1} - 1).$$

Namely, from the center $\mathcal{ZH}(W^{\mathrm{aff}})$ of the affine Hecke algebra, we obtain a commuting family of W-invariant q-difference operators, containing $\mathcal{D}_x$ as a member. All the members of this commuting family are simultaneously diagonalized by the Koornwinder polynomials:

$$\mathcal{D}_f P_\lambda(x) = f(Y) P_\lambda(x) = P_\lambda(x) f(\alpha t^\rho q^\lambda) \qquad (\lambda \in P^+)$$

for all $f \in \mathbb{K}[\xi^{\pm 1}]^W$, where the eigenvalues are given by the evaluation of f at

$$\xi = \alpha t^\rho q^\lambda = (\alpha t^{m-1} q^{\lambda_1}, \alpha t^{m-2} q^{\lambda_2}, \ldots, \alpha q^{\lambda_m}).$$

For the detail, see Stokman's Laredo lectures (2004).

3 Van Diejen's q-difference operators

Koornwinder's q-difference operator $\mathcal{D}_x$ admits a commuting family of algebraically independent q-difference operators $\mathcal{D}_x^{(1)},\ldots,\ \mathcal{D}_x^{(m)}$ (van Diejen, 1994, 1996). To be more precise, there exists a commuting family of W-invariant q-difference operators $\mathcal{D}_f$, parametrized by W-invariant Laurent polynomials $f(\xi) \in \mathbb{K}[\xi^{\pm 1}]^W$ in the dual variables $\xi = (\xi_1,\ldots,\xi_m)$, that are diagonalized by the Koornwinder polynomials $P_\lambda(x)$ ($\lambda \in P^+$):

$$\mathcal{D}_f\, P_\lambda(x) = P_\lambda(x)\, f(\alpha t^\rho q^\lambda), \quad \alpha t^\rho q^\lambda = (\alpha t^{m-1} q^{\lambda_1},\ldots,\alpha t q^{\lambda_{m-1}},\alpha q^{\lambda_m}).$$

In this commuting family, van Diejen's q-difference operators correspond to a certain system of generators of the $\mathbb{K}$-algebra of invariants $\mathbb{K}[\xi^{\pm 1}]^W$. In order to describe van Diejen's operators, we first specify the corresponding W-invariant Laurent polynomials in $\mathbb{K}[\xi^{\pm 1}]^W$.

We introduce the notation

$$\langle z; w\rangle = z + z^{-1} - w - w^{-1} = \langle z/w\rangle\langle zw\rangle$$

corresponding to $\cos(2\pi\zeta)-\cos(2\pi\omega) = -\sin(\pi(\zeta-\omega))\sin(\pi(\zeta+\omega))$ in the additive variables such that $z = e(\zeta)$, $w = e(\omega)$. Note also the eigenvalues d_λ of $\mathcal{D}_x$ is expressed as

$$\begin{aligned} d_\lambda &= \textstyle\sum_{i=1}^m \left(\alpha t^{m-i} q^{\lambda_i} + \alpha^{-1} t^{-m+i} q^{-\lambda_i} - \alpha t^{m-i} - \alpha^{-1} t^{-m+i}\right) \\ &= \textstyle\sum_{i=1}^m \langle \alpha t^{m-i} q^{\lambda_i};\ \alpha t^{m-i}\rangle. \end{aligned}$$

3.1 Fundamental invariants $e_r(\xi;\alpha|t)$ $(r = 1,\ldots,m)$

We define W-invariant Laurent polynomials $e_r(\xi;\alpha|t)$ ($r = 0,1,\ldots,m$) as the expansion coefficients of the product

$$\prod_{j=1}^m \langle u;\xi_j\rangle = \prod_{j=1}^m (u+u^{-1}-\xi_j-\xi_j^{-1}) = \sum_{r=0}^m (-1)^r \langle u;\alpha\rangle_{t,m-r} e_r(\xi;\alpha|t)$$

in terms of the t-shifted factorials associated with $\langle u;\alpha\rangle$:

$$\langle u;\alpha\rangle_{t,k} = \langle u;\alpha\rangle\langle u;\alpha t\rangle\cdots\langle u;\alpha t^{k-1}\rangle = \prod_{i=0}^{k-1}(u + u^{-1} - \alpha t^i - \alpha^{-1} t^{-i}).$$

Then we have

$$\begin{aligned}e_r(\xi;\alpha|t) &= \sum_{1\le j_1<\ldots<j_r\le m} \langle \xi_{j_1};\alpha t^{j_1-1}\rangle\cdots\langle \xi_{j_r};\alpha t^{j_r-r}\rangle \\ &= m_{(1^r)}(\xi) + \text{lower terms w.r.t. the dominance ordering}\end{aligned}$$

for $r=0,1,\ldots,m$. Hence $e_r(\xi;\alpha|t)$ $(r=1,\ldots,m)$ form a generator system of the $\mathbb{K}$-algebra $\mathbb{K}[\xi^{\pm 1}]^W$ of W-invariants. Furthermore, it turns out that $e_r(\xi;\alpha|t)$ is essentially the BC_m interpolation polynomial of Okounkov attached to the fundamental weight (1^r):

(1) For any partition μ with $l(\mu)<r$, i.e. $\mu\not\supset(1^r), e_r(\alpha t^\rho q^\mu;\alpha|t)=0$.

(2) $e_r(\alpha t^\rho q^{1^r};\alpha|t)=(-1)^r\langle\alpha t^{m-r};\alpha t^{m-r}q\rangle_{t,r}$.

3.2 Van Diejen's q-difference operators

Van Diejen's q-difference operators $\mathcal{D}_x^{(1)},\ldots,\mathcal{D}_x^{(m)}$ are characterized as W-invariant q-difference operators corresponding to the fundamental invariants $e_r(\xi;\alpha|t)\in\mathbb{K}[\xi^{\pm}]^W$ $(r=1,\ldots,m)$ introduced above. Namely they are diagonalized by the Koornwinder polynomials as

$$\mathcal{D}_x^{(r)}\,P_\lambda(x)=P_\lambda(x)\,e_r(\alpha t^\rho q^\lambda;\alpha|t)\qquad(\lambda\in P^+).$$

Noting that

$$\prod_{i=1}^m\langle u;\xi_i\rangle=\sum_{r=0}^m(-1)^r\langle u;\alpha\rangle_{t,m-r}\,e_r(\xi;\alpha|t),$$

we introduce the generating function

$$\mathcal{D}_x(u)=\sum_{r=0}^m(-1)^r\langle u;\alpha\rangle_{t,m-r}\,\mathcal{D}_x^{(r)},$$

so that

$$\mathcal{D}_x(u)P_\lambda(x)=P_\lambda(x)\prod_{i=1}^m\langle u;\alpha t^{m-i}q^{\lambda_i}\rangle\qquad(\lambda\in P^+).$$

For each $r=0,1,\ldots,m$, van Diejen's q-difference operator $\mathcal{D}_x^{(r)}$ is expressed in the form

$$\mathcal{D}_x^{(r)}=\sum_{\mu\in P;\,\mu\le(1^r)}A_\mu^{(r)}(x)\,T_{q,x}^\mu$$

with certain rational functions $A^{(r)}_\mu(x)$. Noting that each $\mu \le (1^r)$ can be expressed as $\mu = \sum_{i\in I} \pm\varepsilon_i$ for some subset $I \subset \{1,\ldots,m\}$ with $|I| \le r$, we represent such a μ as a pair (I,ϵ) of a subset I and a mapping $\epsilon : I \to \{\pm 1\}$:

$$\mathcal{D}^{(r)}_x = \sum_{(I,\epsilon);\,|I|\le r} A^{(r)}_{I,\epsilon}(x)\, T^{(I,\epsilon)}_{q,x}.$$

We remark that $\mathcal{D}^{(0)}_x = 1$ and $\mathcal{D}^{(r)}_x(1) = 0$ for $r = 1,\ldots,m$. Namely, for $r > 0$ the constant function 1 is an eigenfunction of $\mathcal{D}^{(r)}_x$ with eigenvalue 0. This means that the constant term $A^{(r)}_\phi(x)$ of $\mathcal{D}^{(r)}_x$ is determined from the other terms by

$$A^{(r)}_\phi(x) = -\sum_{(I,\epsilon);\,0<|I|\le r} A^{(r)}_{I,\epsilon}(x).$$

Setting $M = \{1,\ldots,m\}$, for two subsets $I, J \subset M$ with $I \cap J = \phi$, we define

$$A(x_I; x_J) = \prod_{i\in I} \frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle} \prod_{i,j\in I; i<j} \frac{\langle tx_ix_j\rangle\langle qtx_ix_j\rangle}{\langle x_ix_j\rangle\langle qx_ix_j\rangle} \prod_{i\in I; j\in J} \frac{\langle tx_i/x_j\rangle\langle tx_ix_j\rangle}{\langle x_i/x_j\rangle\langle x_ix_j\rangle}.$$

Note that this function $A(x_I; x_{M\setminus I})$ for $I = \{1,\ldots,r\}$ essentially comes from the positive part $w_+(x)$ of the weight function:

$$A(x_1,\ldots,x_r; x_{r+1},\ldots,x_m) = \text{const.}\frac{T_{q,x_1}\cdots T_{q,x_r} w_+(x)}{w_+(x)}$$

Then, for each subset $I \subset M$ with $|I| = r$, the coefficient $A^{(r)}_{I,\epsilon}(x)$ is given by

$$A^{(r)}_{I,\epsilon}(x) = A(x_I^\epsilon; x_{M\setminus I}),$$

and for $I \subset M$ with $|I| < r$, $A^{(r)}_{I,\epsilon}(x)$ is expressed as the product

$$A^{(r)}_{I,\epsilon}(x) = A(x_I^\epsilon; x_{M\setminus I}) A^{(r-|I|)}_\phi(x_{M\setminus I})$$

with the 0th order term of the operator $\mathcal{D}^{(r-|I|)}_{x_{M\setminus I}}$.

It is also known that the 0th order term of $\mathcal{D}^{(r)}_x$ is expressed as

$$A^{(r)}_\phi(x) = (-1)^r \sum_{(I,\epsilon);|I|=r} B(x_I^\epsilon; x_{M\setminus I}),$$

where

$$B(x_I; x_J) = \prod_{i\in I} \frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle} \prod_{i,j\in I; i<j} \frac{\langle tx_ix_j\rangle\langle qx_ix_j/t\rangle}{\langle x_ix_j\rangle\langle qx_ix_j\rangle} \prod_{i\in I; j\in J} \frac{\langle tx_i/x_j\rangle\langle tx_ix_j\rangle}{\langle x_i/x_j\rangle\langle x_ix_j\rangle}.$$

3.3 Duality and Pieri formula

The values of the Koornwinder polynomials $P_\lambda(x)$ at the points

$$x = at^\rho q^\mu = (at^{m-1}q^{\mu_1}, at^{m-2}q^{\mu_2}, \ldots, aq^{\mu_m}) \qquad (\mu \in P^+)$$

have a remarkable duality property. We denote by $^\circ$ the involutive automorphism of the coefficient field $\mathbb{K}$ determined by $a^\circ = \alpha$, $b^\circ = \beta$, $c^\circ = \gamma$, $d^\circ = \delta$ where

$$\alpha = \sqrt{abcd/q}, \quad \beta = \sqrt{qab/cd}, \quad \gamma = \sqrt{qac/bd}, \quad \delta = \sqrt{qad/bc}.$$

Note that $\alpha\beta = ab$, $\alpha\gamma = ac$, $\alpha\delta = ad$. Also, for each Laurent polynomial $F(x) \in \mathbb{K}[x^\pm]$, we denote by $F^\circ(x)$ the Laurent polynomial obtained by applying the involution $^\circ$ to its coefficients. Then we have

$$\frac{P_\lambda(at^\rho q^\mu)}{P_\lambda(at^\rho)} = \frac{P^\circ_\mu(\alpha t^\rho q^\lambda)}{P^\circ_\mu(\alpha t^\rho)} \qquad (\lambda, \mu \in P^+).$$

If we use the normalization $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(at^\rho)$, this duality implies

$$\widetilde{P}_\lambda(at^\rho q^\mu) = \widetilde{P}^\circ_\mu(\alpha t^\rho q^\lambda) \qquad (\lambda, \mu \in P^+).$$

Combining this duality with the explicit formulas for van Diejen's q-difference operators, we obtain the *Pieri formula* with respect the multiplication by the fundamental invariants $e_r(x; a|t)$ $(r = 0, 1, \ldots, m)$. In fact from

$$\sum_{\nu \le (1^r)} A^{(r)}_\nu(x) \widetilde{P}_\lambda(q^\nu x) = \widetilde{P}_\lambda(x)\, e_r(\alpha t^\rho q^\lambda; \alpha|t)$$

we obtain

$$\sum_{\nu \le (1^r)} A^{(r)}_\nu(at^\rho q^\mu) \widetilde{P}_\lambda(at^\rho q^{\mu+\nu}) = \widetilde{P}_\lambda(at^\rho q^\mu)\, e_r(\alpha t^\rho q^\lambda; \alpha|t)$$

by setting $x = at^\rho q^\mu$ $(\mu \in P^+)$. Hence by the duality we have

$$\sum_{\nu \le (1^r)} A^{(r)}_\nu(at^\rho q^\mu) \widetilde{P}^\circ_{\mu+\nu}(\alpha t^\rho q^\lambda) = \widetilde{P}^\circ_\mu(\alpha t^\rho q^\lambda)\, e_r(\alpha t^\rho q^\lambda; \alpha|t).$$

By applying $^\circ$ to this formula, we obtain

$$\sum_{\nu \le (1^r)} A^{(r)\circ}_\nu(\alpha t^\rho q^\mu) \widetilde{P}_{\mu+\nu}(at^\rho q^\lambda) = \widetilde{P}_\mu(at^\rho q^\lambda)\, e_r(at^\rho q^\lambda; a|t) \qquad (\lambda \in P^+)$$

and hence by replacing x for $at^\rho q^\lambda$

$$\sum_{\nu\le(1^r)} A_\nu^{(r)\circ}(\alpha t^\rho q^\mu)\widetilde{P}_{\mu+\nu}(x)=\widetilde{P}_\mu(x)\,e_r(x;a|t).$$

This implies the Pieri formula

$$e_r(x;a|t)\,P_\mu(x)=\sum_{\lambda-\mu\le(1^r)} C_{\lambda/\mu}^{(r)}P_\lambda(x),\quad C_{\lambda/\mu}^{(r)}=A_{\lambda-\mu}^{(r)\circ}(\alpha t^\rho q^\mu)\frac{P_\mu(at^\rho)}{P_\lambda(at^\rho)}.$$

for $r=0,1,\dots,m$. Since $C_{\mu+(1^r)/\mu}^{(r)}=1$, we also have

$$P_{\mu+(1^r)}(at^\rho)=A_{\lambda-\mu}^{(r)\circ}(\alpha t^\rho q^\mu)P_\mu(at^\rho).$$

From this one can evaluate $P_\lambda(x)$ at the reference point at^ρ as follows:

$$\begin{aligned}P_\lambda(at^\rho)=&\prod_{i=1}^m\frac{\langle abcdq^{-1}t^{m-i}\rangle_{q,\lambda_i}\langle abt^{m-i}\rangle_{q,\lambda_i}\langle act^{m-i}\rangle_{q,\lambda_i}\langle adt^{m-i}\rangle_{q,\lambda_i}}{\langle abcdq^{-1}t^{2(m-i)}\rangle_{q,2\lambda_i}}\\&\cdot\prod_{1\le i<j\le m}\frac{\langle t^{j-i+1}\rangle_{q,\lambda_i-\lambda_j}\langle abcdq^{-1}t^{2m-i-j+1}\rangle_{q,\lambda_i+\lambda_j}}{\langle t^{j-i}\rangle_{q,\lambda_i-\lambda_j}\langle abcdq^{-1}t^{2m-i-j}\rangle_{q,\lambda_i+\lambda_j}}.\end{aligned}$$

3.4 Relation to the affine Hecke algebra

The fundamental W-invariants $e_r(\xi;\alpha|t)$ $(r=1,\dots,m)$ naturally arise in the framework of affine Hecke algebra. In the context of q-Dunkl operators, van Diejen's q-difference operators $\mathcal{D}_x^{(r)}$ arise from the q-Dunkl operators $e_r(Y;\alpha|t)$ by restriction to $\mathbb{K}[x^{\pm1}]^W$:

$$\mathcal{D}_x^{(r)}=e_r(Y;\alpha|t)\big|_{\mathbb{K}[x^{\pm1}]^W}\qquad(r=0,1,\dots,m).$$

Also, the generating function $\mathcal{D}_x(u)=\sum_{r=0}^m(-1)^r\langle u;\alpha\rangle_{t,m-r}\mathcal{D}_x^{(r)}$ is expressed as

$$\mathcal{D}_x(u)=\langle u;Y_1\rangle\cdots\langle u;Y_m\rangle\big|_{\mathbb{K}[x^{\pm1}]^W}.$$

Let

$$\mathcal{U}=\sum_{w\in W}t_w^{\frac12}T_w\in\mathcal{H}(W)\tag{1}$$

be the symmetrizer of the Hecke algebra of type C_m, where t_w is defined as $t_w=t_{i_1}\cdots t_{i_l}$ by taking any reduced decomposition $w=s_{i_1}\cdots s_{i_l}$. An interesting fact is that the fundamental W-invariants $e_r(x;a|t)$ with base point a are obtained by applying this symmetrizer to a simple polynomial:

$$\mathcal{U}\big((x_1-a^{-1}t^{-m+r})\cdots(x_r-a^{-1}t^{-m+r})\big) = \mathrm{const}.e_r(x;a|t) \quad (r=0,1,\ldots,m).$$

The constant factor on the right side is the Poincare series of the stabilizer $\mathcal{H}(A_{r-1}\times C_{m-r})$. By applying the Cherednik involution that exchanges x_i and Y_i^{-1}, we also have

$$\mathcal{U}\big((Y_1^{-1}-\alpha^{-1}t^{-m+r})\cdots(Y_r^{-1}-\alpha^{-1}t^{-m+r})\big) = \mathrm{const}.e_r(Y;\alpha|t) \quad (r=0,1,\ldots,m).$$

Note that, on the coefficients field $\mathbb{K}$, Cherednik's involution reduces to the involution $^\circ$ which exchanges (a,b,c,d) with $(\alpha,\beta,\gamma,\delta)$; in terms of the parameters (t_0,t_m,u_0,u_m), $^\circ$ exchanges t_0 and u_m, and t_m, u_0 remain invariant.

By tracing this symmetrization procedure, one can compute the action of $e_r(Y;\alpha|t)$ on W-invariant functions to derive an explicit formula for $\mathcal{D}_f$ with $f=e_r(\xi;\alpha|t)$. In this way one can derive van Diejen's formula for his commuting family of q-difference operators in the framework the affine Hecke algebra.

4 Kernel functions of type BC_m

4.1 Mimachi's kernel function

It is proved by Mimachi (2001) that the function

$$\Psi(x;y) = \prod_{j=1}^{m}\prod_{l=1}^{n}\langle x_j; y_l\rangle = \prod_{j=1}^{m}\prod_{l=1}^{n}(x_j + x_j^{-1} - y_l - y_l^{-1}).$$

satisfies the functional equation

$$\langle t\rangle \mathcal{D}_x \Psi(x;y) + \langle q\rangle \widehat{\mathcal{D}}_y \Psi(x;y) = \langle t^m\rangle\langle q^n\rangle\langle abcdt^{m-1}q^{n-1}\rangle\Psi(x;y)$$

where $\widehat{\mathcal{D}}_y = \mathcal{D}_y^{(a,b,c,d|t,q)}$ is the Koornwinder operator in the y variables with (q,t) replaced by (t,q). From this formula, he established the expansion formula

$$\Psi(x;y) = \sum_{\lambda\subset(n^m)} (-1)^{|\lambda^*|}\, P_\lambda(x;a,b,c,d|q,t)\, P_{\lambda^*}(y;a,b,c,d|t,q)$$

of *dual Cauchy type*, as well as integral representations of Selberg type for Koornwinder polynomials.

4.2 Kernel function of Cauchy type

Kernel functions of *Cauchy type* for type BC_m were discovered recently by Ruijsenaars (2005) and Komori-Noumi-Shiraishi (2009).

Let $\Phi(x;y|q,t)$ be any solution of the following system of first order q-difference equations:

$$T_{q,x_i}\Phi(x;y|q,t) = \Phi(x;y|q,t)\prod_{l=1}^{n}\frac{\langle q^{\frac12}t^{-\frac12}x_i;y_l\rangle}{\langle q^{\frac12}t^{\frac12}x_i;y_l\rangle} \qquad (i=1,\ldots,m),$$
$$T_{q,y_k}\Phi(x;y|q,t) = \Phi(x;y|q,t)\prod_{j=1}^{m}\frac{\langle q^{\frac12}t^{-\frac12}y_k;x_j\rangle}{\langle q^{\frac12}t^{\frac12}y_k;x_j\rangle} \qquad (k=1,\ldots,n).$$

Then $\Phi(x;y|q,t)$ satisfies the functional equation

$$\langle t\rangle\mathcal{D}_x\Phi(x;y|q,t) - \langle t\rangle\widetilde{\mathcal{D}}_y\Phi(x;y|q,t) = \langle t^m\rangle\langle t^{-n}\rangle\langle abcdq^{-1}t^{m-n-1}\rangle\Phi(x;y|q,t),$$

where $\widetilde{\mathcal{D}}_y$ is the Koornwinder operator with parameters (a,b,c,d) replaced by

$$(\sqrt{qt}/a, \sqrt{qt}/b, \sqrt{qt}/c, \sqrt{qt}/d).$$

Furthermore, such a function $\Phi(x;y|q,t)$ intertwines the whole commuting family of van Diejen's q-difference operators. In terms of the generating function

$$\mathcal{D}_x(u)=\sum_{r=0}^{m}(-1)^r\langle u;\alpha\rangle_{t,m-r}\mathcal{D}_x^{(r)}$$

and

$$\widetilde{\mathcal{D}}_y(u)=\sum_{s=0}^{n}(-1)^s\langle u;\widetilde{\alpha}\rangle_{t,n-s}\widetilde{\mathcal{D}}_y^{(s)},\qquad \widetilde{\alpha}=t/\alpha,$$

one has

$$\mathcal{D}_x(u)\Phi(x;y|q,t)=\langle u;\alpha\rangle_{t,m-n}\widetilde{\mathcal{D}}_y(u)\Phi(x;y|q,t)$$

for $m\geq n$. Note that the q-Saalschütz formula implies

$$\langle u;a\rangle_{t,l}=\sum_{r=0}^{l}\frac{\langle t^{-l}\rangle_{t,r}}{\langle t\rangle_{t,r}}\langle t^{l-r}ab\rangle_{t,r}\langle b/a\rangle_{t,r}\langle u;b\rangle_{t,l-r}.$$

Through this change of reference points, we obtain

$$\mathcal{D}_x^{(r)}\Phi(x;y|q,t)=\sum_{p=0}^{r}\frac{(-1)^p}{\langle t\rangle_{t,p}}\langle t^{-n+r-p},t^{m-r+1},t^{-m+n+1}\alpha^{-2}\rangle_{t,p}\widetilde{\mathcal{D}}_y^{(r-p)}\Phi(x;y|q,t).$$

The system of first order difference equations to be satisfied by $\Phi(x;y|q,t)$ determines the kernel function only up to a multiplicative factor that is q-periodic with respect to all the variables (x,y). By choosing such a factor appropriately, one can construct several kernel functions of Cauchy type with different analytic properties. Two typical choices are given by

$$\Phi_0(x;y|q,t)=(x_1\cdots x_m)^{n\kappa}\prod_{j=1}^{m}\prod_{l=1}^{n}\prod_{\epsilon=\pm1}\frac{(q^{\frac{1}{2}}t^{\frac{1}{2}}x_jy_l^{\epsilon};q)_\infty}{(q^{\frac{1}{2}}t^{-\frac{1}{2}}x_jy_l^{\epsilon};q)_\infty}$$

with κ such that $t=q^{\kappa}$, and

$$\Phi_+(x;y|q,t)=g(x;y)\prod_{j=1}^{m}\prod_{l=1}^{n}\prod_{\epsilon_1,\epsilon_2=\pm1}(q^{\frac{1}{2}}t^{\frac{1}{2}}x_j^{\epsilon_1}y_l^{\epsilon_2};q)_\infty,$$

where $g(x;y)$ is an arbitrary function satisfying

$$\begin{aligned}T_{q,x_i}g(x;y)&=g(x;y)(qx_i^2)^n\quad(i=1,\ldots,m),\\ T_{q,y_k}g(x;y)&=g(x;y)(qy_k^2)^m\quad(k=1,\ldots,n).\end{aligned}$$

5 Application of kernel functions

5.1 An explicit formula for Koornwinder polynomials

By combining the Pieri formula

$$e_r(x;a|t)\,P_\mu(x) = \sum_{\lambda-\mu\leq(1^r)} C^{(r)}_{\lambda/\mu} P_\lambda(x), \quad C^{(r)}_{\lambda/\mu} = A^{(r)\circ}_{\lambda-\mu}(\alpha t^\rho q^\mu)\frac{P_\mu(at^\rho)}{P_\lambda(at^\rho)}.$$

with the dual Cauchy formula

$$\Psi(x;y) = \prod_{j=1}^{m}\prod_{l=1}^{n}\langle x_j;y_l\rangle = \sum_{\lambda\subset(n^m)}(-1)^{|\lambda^*|}\,\widehat{P}_\lambda(x)\,P_{\lambda^*}(y)$$

we obtain an explicit formula for Koornwinder polynomials. (Here $\widehat{P}_\lambda(x)$ stands for $P_\lambda(x;a,b,c,d|t,q)$.) In fact, for $x' = (x_1,\ldots,x_{m-1})$ we have

$$\begin{aligned}
\Psi(x;y) &= \Psi(x';y)\prod_{l=1}^{n}\langle x_m;y_l\rangle = \Psi(x';y)\sum_{r=0}^{n}(-1)^r\langle x_m;a\rangle_{t,n-r}e_r(y;a|t) \\
&= \sum_{\mu\subset(n^{m-1})}\sum_{r=0}^{n}(-1)^{|\mu^*|+r}\widehat{P}_\mu(x)\langle x_m;a\rangle_{t,n-r}P_{\mu^*}(y)e_r(y;a|t) \\
&= \sum_{\lambda\subset(n^m)}\sum_{\mu\subset(n^{m-1})}\sum_{r=0}^{n}(-1)^{|\mu^*|+r}\widehat{P}_\mu(x')\langle x_m,a\rangle_{t,n-r}C^{(r)}_{\lambda^*/\mu^*}P_{\lambda^*}(y).
\end{aligned}$$

Hence we have

$$(-1)^{|\lambda^*|}\widehat{P}_\lambda(x) = \sum_{\mu\subset(n^{m-1})}\sum_{0\leq r\leq n}(-1)^{|\mu^*|+r}\widehat{P}_\mu(x')\,C^{(r)}_{\lambda^*/\mu^*}\langle x_m;a\rangle_{t,n-r},$$

namely

$$(-1)^{|\lambda^*|}P_\lambda(x) = \sum_{\mu\subset(n^{m-1})}\sum_{0\leq k\leq n}(-1)^{|\mu^*|+n-k}P_\mu(x')\;\widehat{C}^{(n-k)}_{\lambda^*/\mu^*}\langle x_m;a\rangle_{q,k}.$$

By repeating this procedure, we obtain an explicit formula of the form

$$P_\lambda(x) = \sum_{k_1,k_2,\ldots,k_m} c^\lambda_{k_1,\ldots,k_m}\langle x_1;a\rangle_{q,k_1}\cdots\langle x_m;a\rangle_{q,k_m},$$

where the coefficients are given by

$$c^\lambda_{k_1,\ldots,k_m} = (-1)^{|\lambda|+\sum_{i=1}^m k_i}\sum_{\mu^{(1)},\ldots,\mu^{(m)}}\prod_{i=1}^{m}\widehat{C}^{(n-k_i)}_{\mu^{(i)*}/\mu^{(i-1)*}}$$

summed over all sequences of partitions $\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(m)}$ such that $\mu^{(i)} \subset (n^i)$ $(i = 1, \ldots, m)$ and $\mu^{(0)} = \phi, \mu^{(m)} = \lambda$. This is a generalization of the tableau representation for the Macdonald polynomials of type A_{m-1} to the BC_m case.

5.2 Koornwinder polynomials attached to single columns

The Koornwinder polynomials attached to single columns are obtained as the expansion coefficients of

$$\Psi(x; y) = \prod_{j=1}^{m} \langle x_j; y \rangle = \sum_{r=0}^{m} (-1)^{m-r} P_{(1^r)}(x)\, p_{m-r}(y; t)$$

in terms of the Askey-Wilson polynomials $p_{m-r}(y; t)$ with base t. By using the ${}_4\phi_3$-expression of the Askey-Wilson polynomials, one can derive an expression of $P_{(1^r)}(x)$ in terms of the fundamental W-invariants $e_l(x; a|t)$ $(l = 0, 1, \ldots, m)$.

Recall the fundamental W-invariant polynomials $e_r(x; a|t)$ $(r = 0, 1, \ldots, m)$ are expressed as

$$e_r(x; a|t) = \sum_{1 \le j_1 < \ldots < j_r \le m} \langle x; at^{j_1 - 1} \rangle \cdots \langle x; at^{j_r - r} \rangle \qquad (r = 0, 1, \ldots, m).$$

The Koornwinder polynomials $P_{(1^r)}(x)$ attached to single columns are expanded as follows in terms of $e_l(x; a|t)$:

$$P_{(1^r)}(x) = \sum_{l=0}^{r} \frac{\langle t^{m-r+1}, t^{m-r}ab, t^{m-r}ac, t^{m-r}ad, \rangle_{t,l}}{\langle t, t^{2(m-r)}abcd \rangle_{t,l}}\, e_{r-l}(x; a|t).$$

5.3 Koornwinder polynomials attached to single rows

When $t = q^{-k}$ $(k = 0, 1, 2, \ldots)$, the kernel function $\Phi_0(x; y|q, t)$ of Cauchy type reduces to Laurent polynomials

$$\Phi_0(x; y|q, q^{-k}) = (-1)^{kmn} \prod_{j=1}^{m} \prod_{l=1}^{n} \langle y_l; q^{\frac{1}{2}(1-k)} x_j \rangle_{q,k}.$$

By using the expansion of these functions for $n = 1$ in terms of Askey-Wilson polynomials, one can also derive a new explicit formula for Koornwinder polynomials attached to single rows.

In order to describe Koornwinder polynomials attached to single rows, we introduce the Laurent polynomials $h_l(x;a|q,t)$ $(l=0,1,2,\ldots)$ by

$$h_l(x;a|q,t)=\sum_{\nu_1+\cdots+\nu_m=l}\prod_{i=1}^{m}\frac{\langle t\rangle_{q,\nu_i}}{\langle q\rangle_{q,\nu_i}}\langle x_i;at^{i-1}q^{\sum_{1\le j<i}\nu_j}\rangle_{q,\nu_i}.$$

In spite of the appearance, these Laurent polynomials are W-invariant, and essentially coincide with the BC_m interpolation polynomials of Okounkov attached single rows. The Koornwinder polynomials $P_{(r)}(x)$ attached to single columns are then expressed as follows in terms of $h_l(x;a|t)$:

$$\begin{aligned}\frac{\langle t\rangle_{q,r}}{\langle q\rangle_{q,r}}P_{(r)}(x)=&\frac{\langle t^m,t^{m-1}ab,t^{m-1}ac,t^{m-1}ad\rangle_{q,r}}{\langle q,t^{2(m-1)}abcdq^{r-1}\rangle_{q,r}}\\&\cdot\sum_{l=0}^{r}\frac{(-1)^l\langle q^{-r},t^{2(m-1)}abcdq^{r-1}\rangle_{q,l}}{\langle t^m,t^{m-1}ab,t^{m-1}ac,t^{m-1}ad\rangle_{q,l}}\,h_l(x;a|q,t).\end{aligned}$$

This is a natural generalization of the ${}_4\phi_3$-expression of Askey-Wilson polynomials to the BC_m case.

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